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LOW-FREQUENCY SCATTERING FROM TWO-DIMENSIONAL PERFECT CONDUCTORS

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Exact expressions are obtained for the leading terms in the low-frequency expansions of the far field scattered by an arbitrarily shaped cylinder with finite cross section, an arbitrarily shaped cylindrical bump on a ground plane, and an arbitrarily shaped cylindrical dent in a ground plane. By inserting the low-frequency expansions of the incident plane wave and Green's function into exact integral equations for the surface current, integral equations are obtained for the leading terms in the low-frequency expansions of the surface current. Simple integrations of these leading terms of the current expansion yield the leading terms in the low-frequency expansions of the scattered fields. For the cylinder with finite cross section, the leading term in the low-frequency expansion of the TM scattered far field is explicitly given by an expression that is independent of the shape of the cylinder. The explicit expression for the low-frequency TE scattered far field contains three constants that depend only on the shape of the cylinder. These three constants are found from the solutions to two electrostatic problems. The explicit expressions for the low-frequency diffracted fields of a bump or dent contain one constant that depends only on the shape of the bump or dent. Remarkably, this single constant is the same for both TM and TE polarization and can be found from the solution to either an electrostatic or magnetostatic problem. The general low-frequency expressions are confirmed by comparing them to low-frequency results obtained from exact time-harmonic eigenfunction solutions, and constants are evaluated for a number of geometries.

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Chapter 1

Introduction

The study of low-frequency scattering was initiated by Rayleigh [1] in 1897 and today the term "Rayleigh scattering" is often used instead of "low-frequency scattering". Kleinman [2] gives the following definition of Rayleigh scattering:

> *We are dealing with Rayleigh scattering when the far-zone field may be expanded in a convergent series in positive integral powers of the propagation constant k.*

The starting point in a Rayleigh scattering calculation is the expansion of the unknown fields in powers of k and then the determination, from Maxwell's equations and boundary conditions, of the unknown expansion coefficients, which are functions of the geometry of the scatterer and the angles of incidence and observation. This procedure is used by Stevenson [3] and by Asvestas and Kleinman [4] to determine the low-frequency electromagnetic scattering from perfectly electrically conducting three-dimensional bodies.

However, as noted in [2] this definition of Rayleigh scattering cannot be used for two-dimensional scattering problems since the scattered field in general does not have a convergent expansion in powers of k. The two-dimensional fields cannot be expanded in powers of k because the two-dimensional free-space Green's function is a Hankel function which has a branch point at $k=0$, whereas the the three-dimensional free-space Green's function is analytic in k.

In the literature much work on low-frequency scattering from general three-dimensional bodies has been published [2], but it seems that only van Bladel [5] and MacCamy [6] have done work on low-frequency scattering from general two-dimensional bodies. In [5] the low-frequency scattering from arbitrarily shaped dielectric and conducting cylinders with finite cross sections is considered. For transverse magnetic (TM) polarization [5] uses a time-harmonic integral equation to obtain the low-frequency behavior of the total current from which the low-frequency scattered field is calculated. For transverse electric (TE) polarization an expansion of the current in integral powers of k is assumed. As seen from the previous discussion and from the exact eigenfunction solutions for the circular cylinder [7, ch.2] and strip [7, ch.4] such an expansion does not exist. However, only the first two terms in this power series expansion are used in [5] and in this paper it will be shown that the final TE results in [5] for perfect conductors are also correct. Reference [6] contains the formal

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low-frequency expansion of the scattered fields from two-dimensional perfect conductors with finite cross section in cases where the incident field can be expanded in even powers of k. However, no explicit expressions for the expansion coefficients are given.

There are several reasons for being interested in two-dimensional scattering solutions. They can be used directly as approximate solutions to certain three-dimensional scattering problems. For example, the field close to the middle of a long finite rod can often be well approximated by the field from the corresponding infinite rod. Also, two-dimensional scattering solutions can be helpful for validating computer codes.

Another important reason for being interested in two-dimensional solutions is that they can be used to determine three-dimensional incremental length diffraction coefficients. These incremental length diffraction coefficients can in turn be used to determine scattering contributions from, for example, curved ridges (bumps) and channels (dents) that have constant cross sections. The three-dimensional incremental diffraction coefficients can be found directly from the corresponding two-dimensional far fields using a direct substitution approach. No integration, differentiation, or specific knowledge of the current on the conductor is needed. This direct substitution procedure was first developed by Shore and Yaghjian [8] for planar surfaces and then extended by Hansen and Yaghjian [9] to general two-dimensional scatterers. To apply this procedure one must, in general, be able to evaluate the two-dimensional far fields for both real and complex angles of observation. Therefore, one needs analytical expressions for the two-dimensional far fields.

Analytical expressions for two-dimensional far fields from single cylindrical bumps and dents can also be used in conjunction with the work of Twersky [10] to calculate the scattering from a random distribution of bumps or dents.

The purpose of this report is to determine the leading terms in the low-frequency expansions of the scattered electromagnetic far fields for the following three types of two-dimensional perfectly electrically conducting scatterers:

1. An arbitrarily shaped cylinder with finite cross section

2. An arbitrarily shaped cylindrical bump (protuberance) on a ground plane

3. An arbitrarily shaped cylindrical dent (indentation) in a ground plane.

The results of this paper apply to bumps and dents that are continuously lined by a conductor and thus they are not generally valid for bumps and dents that contain slits. The scatterers are illuminated by plane waves propagating in directions normal to the axis of the cylinders, and the two polarization cases, TM and TE, are treated separately. Since the scatterers are perfectly conducting the low-frequency solutions for normal incidence can be immediately generalized to obtain low-frequency solutions for oblique incidence (see, for example, [11, sec.8.15] or [7, ch.1]).

As mentioned above, the unknown field for two-dimensional scatterers, unlike the field for three-dimensional scatterers, cannot be expanded in powers of k. Instead we take the following approach: The Green's functions and incident field in the time-harmonic integral

equation for the current is expanded for small k to obtain an integral equation for the low-frequency current. From this low-frequency integral equation we find the leading terms in the low-frequency expansion of the current and calculate the low-frequency far field.

The low-frequency expressions for the scattered fields from the cylinders with finite cross section are derived in Chapter 2. For the cylindrical bump and dent the scattered fields are written as the sum of a known reflected field and a diffracted field. The low-frequency expressions for the diffracted field from the bump and dent are derived in Chapter 3 and Chapter 4, respectively. In Chapter 5 the low-frequency expressions are verified in cases where exact time-harmonic eigenfunction solutions exist and the constants are calculated for a number of geometries. A summary of this work was first presented in [12].

Chapter 2

Cylinder with finite cross section

The cylinder with finite cross section illuminated by the plane-wave field $(\bar{E}^i, \bar{H}^i)$ is situated in a rectangular coordinate system shown in Figure 2.1. The scatterer extends uniformly to infinity in the $+z$ and $-z$ directions and the curve that describes the finite cross section of the scatterer in the $x-y$ plane is denoted by S. The outward normal to the scatterer is $\hat{n}$ and the tangent unit vector to S is $\hat{t} = \hat{z} \times \hat{n}$. A characteristic dimension of the scatterer in the $x-y$ plane is denoted by d. In addition to the rectangular coordinates (x, y, z), circular cylindrical coordinates (r, ϕ, z) given by $x = r\cos\phi$, $y = r\sin\phi$, and $z = z$ will also be used.

The incident plane-wave field propagates in a direction, designated by ϕ^i, normal to the z-axis. As usual, the scattered field $(\bar{E}^s, \bar{H}^s)$ is defined as the total field $(\bar{E}, \bar{H})$ minus the incident field $(\bar{E}^i, \bar{H}^i)$.

Throughout the report $e^{j\omega t}$ time dependence is suppressed in all the time-harmonic equations.

2.1 Transverse magnetic (TM) polarization

The incident electric field is given by

$$\bar{E}^i(\bar{r}) = \hat{z}e^{jk(x\cos\phi^i + y\sin\phi^i)} \tag{2.1}$$

where k is the propagation constant. The low-frequency current is determined in Section 2.1.1 and in Section 2.1.2 it is integrated to get the low-frequency far field.

2.1.1 Low-frequency current (TM)

To determine the low-frequency far field we first determine the low-frequency current. As we shall see, for the cylinder with finite cross section for TM polarization it is sufficient to know the total current flowing in the $z-$direction on the cylinder. To find this current we use the vector potential for the scattered field

$$\bar{A}(\bar{r}) = \int_S G(\bar{r}, \bar{r}')\bar{K}(\bar{r}')ds' \tag{2.2}$$

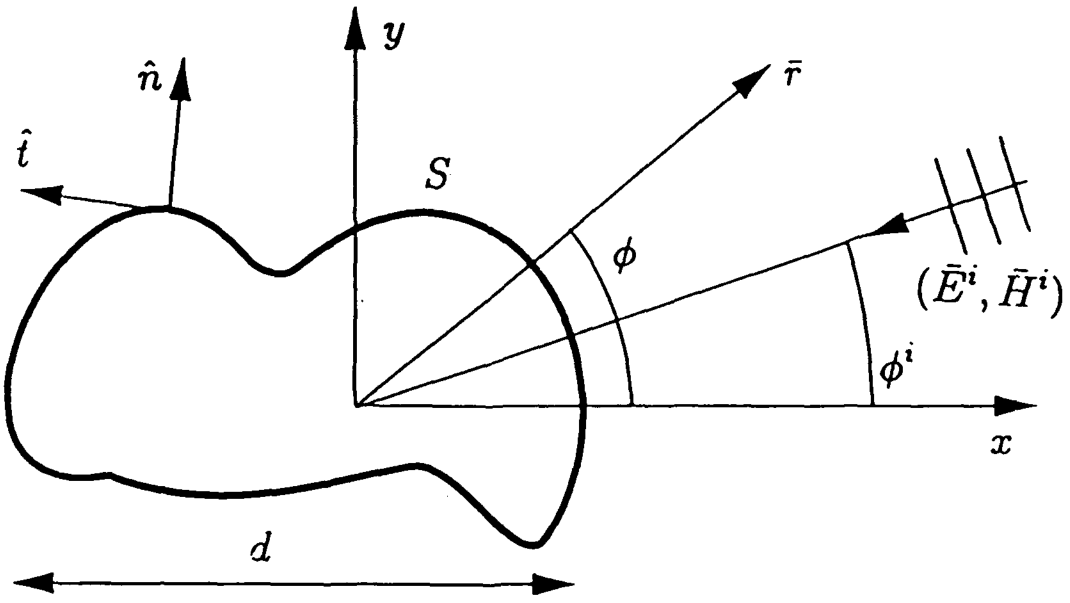

Figure 2.1: Cylinder with finite cross section.

from which the scattered fields are found as

$$\bar{E}^s = -\frac{j}{k}\sqrt{\frac{\mu}{\epsilon}}\nabla \times \nabla \times \bar{A}, \quad \bar{H}^s = \nabla \times \bar{A}. \tag{2.3}$$

Here $G(\bar{r}, \bar{r}')$ is the two-dimensional free-space Green's function

$$G(\bar{r}, \bar{r}') = -\frac{j}{4}H_0^{(2)}(k \mid \bar{r} - \bar{r}' \mid) \tag{2.4}$$

and $\bar{K}(\bar{r}) = \hat{z}K_z(\bar{r})$ is the current on the cylinder. Because the vector potential has only a $z-$component and the free-space Green's function satisfies the scalar homogeneous Helmholtz equation for $\bar{r} \neq \bar{r}'$ we find from (2.2) and (2.4) that

$$E_z^s(\bar{r}) = -jk\sqrt{\frac{\mu}{\epsilon}}\int_S G(\bar{r}, \bar{r}')K_z(\bar{r}')ds' \tag{2.5}$$

which is a relation between the scattered electric field and the current on the cylinder.

Equation (2.5) and the boundary condition of zero tangential electric field on the conductor yield the integral equation [13],[14]

$$E_z^i(\bar{r}) = jk\sqrt{\frac{\mu}{\epsilon}}⨍_S G(\bar{r}, \bar{r}')K_z(\bar{r}')ds', \quad \bar{r} \in S \tag{2.6}$$

where the bar on the integral sign indicates that the singularity at $\bar{r} = \bar{r}'$ is excluded.

From the small argument approximation for the Hankel function we find that for $\bar{r} \neq \bar{r}'$

$$G(\bar{r}, \bar{r}') = -\frac{1}{2\pi}\ln kd + O((kd)^0), \quad kd \to 0 \tag{2.7}$$

where d is a characteristic dimension of the cylinder.

Because the incident electric field in (2.1) for low frequencies is approximately equal to $\hat{z}$ on the circumference S of the scatterer it is assumed that the total current for low frequencies is non-zero, that is,

$$\int_S K_z(\bar{r}')ds' \neq 0. \tag{2.8}$$

This assumption is confirmed by the eigenfunction solutions for the circular cylinder [7, ch.2] and strip [7, ch.4].

Inserting the expansion (2.7) for the Green's function into the integral equation (2.6) and using (2.8) yields the low-frequency behavior of the total current

$$-\ln kd \frac{jk}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \int_S K_z(\bar{r}')ds' \sim 1\ , \quad kd \to 0. \tag{2.9}$$

2.1.2 Low-frequency scattered far field (TM)

We will now find the low-frequency far field by integrating the low-frequency current given in (2.9). From the asymptotic expansion of the Hankel function it is found that

$$G(\bar{r},\bar{r}') \sim \frac{e^{-j\pi/4}}{2\sqrt{2\pi}} \frac{e^{-jkr}}{\sqrt{kr}} [1 + jk(x'\cos\phi + y'\sin\phi) + O((kd)^2)]\ , \quad r \to +\infty. \tag{2.10}$$

When we insert (2.10) into the integral (2.5) for the scattered field we find that the low-frequency far field is given by

$$E_z^s(\bar{r}) \sim \frac{ke^{-j3\pi/4}}{2\sqrt{2\pi}} \sqrt{\frac{\mu}{\epsilon}} \frac{e^{-jkr}}{\sqrt{kr}} \int_S K_z(\bar{r}')ds'. \tag{2.11}$$

Substituting the integral of the current from (2.9) into the field expression (2.11) we find the final low-frequency expression for the far field [5]

$$E_z^s(\bar{r}) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-j\pi/4}}{\ln kd} \frac{e^{-jkr}}{\sqrt{kr}}. \tag{2.12}$$

Note that $\ln kd \sim \ln Akd$ as $kd \to 0$ when A is a positive constant so that d can be any characteristic dimension of the cross section of the scatterer. Furthermore, it is noted that the low-frequency scattered field does not depend on the shape of the finite cross section but only on its characteristic dimension.

The expression (2.12) checks with the low-frequency results for the circular cylinder [7, ch.2] and strip [7, ch.4] obtained from exact eigenfunction solutions.

It should be noted that the expression in (2.12) for the far field cannot be obtained from the solution to the purely electrostatic or magnetostatic problem in which the scatterer is situated in the static field $\bar{E}^i|_{k=0}$ or $\bar{H}^i|_{k=0}$, respectively:

Since the total electrostatic field has only a $z-$component one finds from Maxwell's equations that it must be a constant and therefore zero because it is zero on the surface

of the scatterer. Consequently, the scattered electrostatic field equals the negative of the incident field. The normal component of this electrostatic solution is zero everywhere so that the charge on the conductor is zero. Moreover, this electrostatic solution gives no information about the current on the conductor and thus no information about the scattered time-harmonic fields.

As part of the solution to the magnetostatic equations ($\nabla \times \bar{H} = 0, \nabla \cdot \bar{H} = 0$, and $\hat{n} \cdot \bar{H} = 0$ on S) we can have a total current on the conductor. However, the value of this current is not determined by these magnetostatic equations.

2.2 Transverse electric (TE) polarization

In this section we will determine the low-frequency far field in the case of TE polarization. In Section 2.2.1 we determine a low-frequency expansion of the current and in Section 2.2.2 we integrate this low-frequency expansion to get the far field.

The incident electric and magnetic fields are here given by

$$\bar{H}^i(\bar{r}) = \hat{z}e^{jk(x\cos\phi^i + y\sin\phi^i)} \tag{2.13}$$

and

$$\bar{E}^i(\bar{r}) = \sqrt{\frac{\mu}{\epsilon}}(\sin\phi^i\hat{x} - \cos\phi^i\hat{y})e^{jk(x\cos\phi^i + y\sin\phi^i)}. \tag{2.14}$$

2.2.1 Low-frequency current (TE)

To obtain the low-frequency current we will insert the low-frequency expansions for the Green's function and the incident field into the two-dimensional magnetic field integral equation. We will start by deriving this integral equation. Using the relations (2.2)and (2.3) involving the vector potential and the fact that

$$\nabla \times [G(\bar{r},\bar{r}')\bar{K}(\bar{r}')] = -\hat{z}K_t(\bar{r}')\frac{\partial}{\partial n'}G(\bar{r},\bar{r}') \tag{2.15}$$

where t and n refer to the tangent and normal unit vectors in Figure 2.1, we obtain

$$H_z^s(\bar{r}) = -\int_S K_t(\bar{r}')\frac{\partial}{\partial n'}G(\bar{r},\bar{r}')ds' \tag{2.16}$$

which is a relation between the current and the scattered magnetic field.

We now let $\bar{r}$ in (2.16) approach a point on S and use the result from [14, App.] of an integration near the singularity at $\bar{r}' = \bar{r}$ and the boundary condition $H_z = -K_t$ on S to get the magnetic field integral equation [14]

$$H_z^i(\bar{r}) = \fint_S K_t(\bar{r}')\frac{\partial}{\partial n'}G(\bar{r},\bar{r}')ds' - \frac{1}{2}K_t(\bar{r}), \quad \bar{r} \in S. \tag{2.17}$$

The bar on the integral sign indicates that this is a Cauchy principal value integration. To determine the low-frequency expansion of the current we use the small argument expansion of the Hankel function to show that

$$\frac{\partial}{\partial n'}G(\bar{r},\bar{r}') = \frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}') + O((kd)^2 \ln kd) \tag{2.18}$$

where $G^0(\bar{r},\bar{r}')$ is the static two-dimensional free-space Green's function given by

$$G^0(\bar{r},\bar{r}') = -\frac{1}{2\pi}\ln\frac{|\bar{r}-\bar{r}'|}{d}. \tag{2.19}$$

The term of order $(kd)^2 \ln kd$ in (2.18) is non-singular at $\bar{r} = \bar{r}'$; in fact, it tends to zero as $\bar{r} \longrightarrow \bar{r}'$.

Letting $kd \rightarrow 0$ in the integral equation (2.17) and using the expansion (2.18) we find that the first term K_t^0 in the low-frequency expansion of the current K_t satisfies the integral equation

$$1 = \fint_S K_t^0(\bar{r}')\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')ds' - \frac{1}{2}K_t^0(\bar{r}), \quad \bar{r} \in S. \tag{2.20}$$

The uniqueness of the solution to this integral equation has been proven in [15].

We will now show that the solution to the integral equation (2.20) is related to a magnetostatic problem: A static magnetic field and current satisfy the relation (2.16) with $G(\bar{r},\bar{r}')$ replaced by $G^0(\bar{r},\bar{r}')$. Furthermore, because the second term in the expansion (2.18) of the derivative of the Green's function is non-singular at $\bar{r}' = \bar{r}$ it follows that the integral equation (2.17) holds in the static case. Equation (2.20) is thus the integral equation for the static current when the scatterer is situated in the impressed magnetostatic field

$$\bar{H}^i_{static} = \bar{H}^i|_{k=0} = \hat{z}. \tag{2.21}$$

Let H_z^0 be the solution to this static problem. Then since $\nabla \times \bar{H}^0 = 0$ we find that H_z^0 is a constant. Furthermore, the impressed magnetostatic field in (2.21) satisfies the boundary condition $\hat{n} \cdot \bar{H}^0 = 0$ on the conductor, and since the scattered field must vanish far away from the scatterer it follows that

$$\bar{H}^0 = \bar{H}^i_{static} = \hat{z}. \tag{2.22}$$

We can therefore write the current $K_t(\bar{r})$ as

$$K_t(\bar{r}) = -1 + K_t^r(\bar{r}) \tag{2.23}$$

where $K_t^r(\bar{r}) \rightarrow 0$ as $kd \rightarrow 0$. We have hereby obtained the first term in the low-frequency expansion of the current and we will now find the second term.

If we use the expansion (2.23) of the current, the expansion (2.18) of the Green's function, and (2.20) in the integral equation (2.17) we find that the second term K_t^1 in the low-frequency expansion of the current K_t satisfies the integral equation

$$jk(x\cos\phi^i + y\sin\phi^i) = \fint_S K_t^1(\bar{r}')\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')ds' - \frac{1}{2}K_t^1(\bar{r}), \quad \bar{r} \in S. \tag{2.24}$$

Consequently, the second term is of order kd and we can expand the current as

$$K_t(\bar{r}) = -1 + kdK_t^{01}(\bar{r}) + K_t^{rr}(\bar{r}) \tag{2.25}$$

where $K_t^{01} = \frac{1}{kd}K_t^1$ and $\frac{1}{kd}K_t^{rr} \to 0$ as $kd \to 0$

We will now show that the current K_t^{01}, which is independent of kd, can be found from the solution to the electrostatic problem in which the scatterer is situated in the impressed electrostatic field

$$\bar{E}^i_{static} = \bar{E}^i|_{k=0} = \sqrt{\frac{\mu}{\epsilon}}(\hat{x}\sin\phi^i - \hat{y}\cos\phi^i). \tag{2.26}$$

Let the electrostatic solution be denoted by $\bar{E}^0$. Then $\nabla \cdot \bar{E}^0 = 0$, that is, the electrostatic field is solenoidal. In Appendix A it is shown that because of this and because the total charge on the conductor is zero the field $-E_y^0\hat{x} + E_x^0\hat{y}$ is conservative. We can therefore introduce a scalar potential F_z^0 so that $-E_y^0\hat{x} + E_x^0\hat{y} = \sqrt{\frac{\mu}{\epsilon}}\nabla F_z^0$. Defining $\bar{F}^0 = F_z^0\hat{z}$ one finds that

$$\bar{E}^0 = \sqrt{\frac{\mu}{\epsilon}}\nabla \times \bar{F}^0 \tag{2.27}$$

and the electrostatic field is thereby written in terms of the vector potential $\bar{F}^0$. Because the electrostatic field is irrotational, that is, $\nabla \times \bar{E}^0 = 0$ it follows that F_z^0 satisfies Laplace's equation $\nabla^2 F_z^0 = 0$. Since the tangential component of the electric field is zero on the conductor, F_z^0 satisfies the Neumann boundary condition on the conductor. Hence, F_z^0 satisfies the same differential equation (Laplace's equation) and the same boundary conditions on the conductor (Neumann type) as the $z-$component of a magnetostatic field. If we furthermore require that the potential F_z^{0s} for the scattered electrostatic field tend to zero at infinity we find that F_z^0 satisfies the integral equation (2.17) with H_z^i, G, and K_t replaced by F_z^{0i}, G^0, and $-F_z^0$, respectively. The impressed electrostatic field given in (2.26) corresponds to the impressed vector potential

$$F_z^{0i}(\bar{r}) = x\cos\phi^i + y\sin\phi^i + C \tag{2.28}$$

where C is a constant. The integral equation for F_z^0 is thus

$$x\cos\phi^i + y\sin\phi^i + C = -\int_S F_z^0(\bar{r}')\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')ds' + \frac{1}{2}F_z^0(\bar{r}), \quad \bar{r} \in S. \tag{2.29}$$

Using the result from [14, App.] of the integration near the singularity at $\bar{r}' = \bar{r}$ and the divergence theorem in the region bounded by S one finds that

$$C = -\int_S \frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')Cds' + \frac{1}{2}C, \quad \bar{r} \in S. \tag{2.30}$$

Consequently, the constant function C on the left hand side of the integral equation (2.29) only adds the constant C to the solution to (2.29) with $C = 0$. Because we may add a constant to the potential $\bar{F}^0$ without changing the field $\bar{E}^0$ we can let $C = 0$ in the integral equation (2.29).

By comparing the integral equations (2.29) with $C = 0$ and (2.24) and noting that the solutions are unique one finds $K_t^{01}(\bar{r}) = -\frac{j}{d}F_z^0(\bar{r}), \;\; \bar{r} \in S$, which completes the proof that the second term in the low-frequency expansion of the current can be found from the electrostatic solution. The low-frequency expansion of the current is thus

$$K_t(\bar{r}) = -1 - jkF_z^0(\bar{r}) + K_t^{rr}(\bar{r}) \tag{2.31}$$

where F_z^0 satisfies the electrostatic integral equation (2.29) and $\frac{1}{kd}K_t^{rr} \to 0$ as $kd \to 0$.

In this Section we have shown that the first and second term in the TE low-frequency expansion of the current are of order $(kd)^0$ and kd, respectively. Since the second term in the low-frequency expansion (2.18) of the derivative of the free-space Green's function is of order $(kd)^2 \ln kd$ it is seen from the integral equation (2.17) that the third term in the low-frequency expansion of the current, in general, will be a function of kd and $\ln kd$. Therefore, as noted in the Introduction, the current cannot be expanded in a power series in kd.

2.2.2 Low-frequency scattered far field (TE)

We will now calculate the far field scattered by the current given in the expansion (2.31). Using the asymptotic expansion of the Hankel function one finds that

$$\frac{\partial}{\partial n'}G(\bar{r},\bar{r}') \sim k\frac{e^{j\pi/4}}{2\sqrt{2\pi}}\frac{e^{-jkr}}{\sqrt{kr}}[1 + jk(x'\cos\phi + y'\sin\phi) + O((kd)^2)]\hat{n}'\cdot\hat{r} \tag{2.32}$$

as $r \to +\infty$. Inserting (2.32) and the current expansion (2.31) into the expression (2.16) for the scattered magnetic field one finds that the scattered field to order $(kd)^2$ is given by

$$H_z^s(\bar{r}) \sim -(kd)^2\frac{e^{-j\pi/4}}{2\sqrt{2\pi}}\frac{e^{-jkr}}{\sqrt{kr}}\frac{1}{d^2}[\hat{r}\cdot\int_S(x'\cos\phi + y'\sin\phi)\hat{n}'ds' + \hat{r}\cdot\int_S F_z^0(\bar{r}')\hat{n}'ds']. \tag{2.33}$$

Applying the divergence theorem one finds that the first term in the brackets of (2.33) equals the area bounded by S, that is, the area of the cross section of the scatterer. Integrating the second term in the brackets of (2.33) by parts, and using that

$$\frac{\sigma^0(\bar{r})}{\epsilon} = \hat{n}\cdot\bar{E}^0(\bar{r}) = \sqrt{\frac{\mu}{\epsilon}}\frac{\partial}{\partial s}F_z^0(\bar{r}), \;\; \bar{r} \in S. \tag{2.34}$$

where σ^0 is the electrostatic charge on the conductor, we find that the scattered field is given by

$$H_z^s(\bar{r}) \sim -(kd)^2\frac{e^{-j\pi/4}}{2\sqrt{2\pi}}\frac{e^{-jkr}}{\sqrt{kr}}\frac{1}{d^2}[A_S + \sqrt{\frac{\epsilon}{\mu}}(\hat{r}\times\hat{z})\cdot\int_S\frac{\sigma^0(\bar{r}')}{\epsilon}\bar{r}'ds'] \tag{2.35}$$

where A_S is the area of the cross section of the scatterer. From this expression it is seen that the scattered field consists of a magnetic dipole in the $z-$direction (the term with A_S) plus an electric dipole in the $x - y$ plane (the integral term) [11, Sec.3.8].

This expression for the scattered field may be rewritten in a form that is more convenient for the following section which deals with the cylindrical bump. From the expression (2.26)

for the impressed electrostatic field $\bar{E}^i_{static}$ it follows that the total electrostatic charge σ^0 can be written as

$$\sigma^0 = \sqrt{\frac{\mu}{\epsilon}} \sin\phi^i \sigma^{0x} - \sqrt{\frac{\mu}{\epsilon}} \cos\phi^i \sigma^{0y} \tag{2.36}$$

where σ^{0x} and σ^{0y} are the electrostatic charges in the electrostatic problems where the scatterer is situated in the impressed electrostatic fields $\hat{x}$ and $\hat{y}$, respectively. Using the expression (2.36) for σ^0 and the dipole moment reciprocity theorem

$$\int_S \sigma^{0x}(\bar{r}')y'ds' = \int_S \sigma^{0y}(\bar{r}')x'ds' \tag{2.37}$$

proven in Appendix B the scattered magnetic far field may be written as

$$\begin{aligned} H^s_z(\bar{r}) \sim -(kd)^2 \frac{e^{-j\pi/4}}{2\sqrt{2\pi}} \frac{e^{-jkr}}{\sqrt{kr}} \frac{1}{d^2} [A_S + C_1 \sin\phi \sin\phi^i \\ + C_2 \cos\phi \cos\phi^i + C_3 \sin(\phi + \phi^i)] \end{aligned} \tag{2.38}$$

where

$$C_1 = \int_S \frac{\sigma^{0x}(\bar{r})}{\epsilon} x ds, \quad C_2 = \int_S \frac{\sigma^{0y}(\bar{r})}{\epsilon} y ds, \quad C_3 = -\int_S \frac{\sigma^{0y}(\bar{r})}{\epsilon} x ds. \tag{2.39}$$

We will now present integral equations that determine the electrostatic charges σ^{0x} and σ^{0y}. An integral equation for σ^{0x} can be derived from the scalar potential ψ for the electrostatic field. The potential ψ satisfies Laplace's equation and the potential for the impressed electrostatic field is given by $\psi^i = -x + C$, where C is a constant. Noting that the potential for the scattered field is given by $\psi^s = \int_S G^0 \sigma^{0x}/\epsilon ds'$ and that the total potential is constant on S, one obtains an integral equation involving an unknown constant. This constant is evaluated at an arbitrary observation point $\bar{r}_0 = (x_0, y_0) \in S$ so that σ^{0x} satisfies the integral equation

$$⨍_S [G^0(\bar{r}, \bar{r}') - G^0(\bar{r}_0, \bar{r}')] \frac{\sigma^{0x}(\bar{r}')}{\epsilon} ds' = x - x_0, \quad \bar{r} \in S. \tag{2.40}$$

In [15] it is shown that the integral equation (2.40) along with the condition that the total charge on the conductor is zero, that is, $\int_S \sigma^{0x} ds = 0$, determine the electrostatic charge σ^{0x} uniquely. Similarly, σ^{0y} is determined by

$$⨍_S [G^0(\bar{r}, \bar{r}') - G^0(\bar{r}_0, \bar{r}')] \frac{\sigma^{0y}(\bar{r}')}{\epsilon} ds' = y - y_0, \quad \bar{r} \in S. \tag{2.41}$$

along with the condition that the total charge on the conductor is zero, that is, $\int_S \sigma^{0y} ds = 0$.

Alternatively, the three constants C_1, C_2, and C_3 in (2.39) can be written in terms of vector potentials for the electrostatic fields. From the integral equation (2.29) with $C = 0$ it is seen that we may introduce vector potentials $\bar{F}^{0x}$ and $\bar{F}^{0y}$ satisfying

$$x = -⨍_S \frac{\partial}{\partial n'} G^0(\bar{r}, \bar{r}') F^{0x}_z(\bar{r}') ds' + \frac{1}{2} F^{0x}_z(\bar{r}), \quad \bar{r} \in S. \tag{2.42}$$

and

$$y = -\int_S \frac{\partial}{\partial n'} G^0(\bar{r}, \bar{r}') F_z^{0y}(\bar{r}') ds' + \frac{1}{2} F_z^{0y}(\bar{r}), \quad \bar{r} \in S \tag{2.43}$$

so that

$$F_z^0 = \cos\phi^i F_z^{0x} + \sin\phi^i F_z^{0y}. \tag{2.44}$$

The integral equations (2.42) and (2.43) are of the same form as the time-harmonic two-dimensional magnetic field integral equation and they should therefore be easy to solve numerically when no part of the scatterer is flat [16, p.168]. (For flat scatterers, (2.40) and (2.41) can be used.) Using the relation (2.34) between σ^0 and F_z^0 and the relations (2.36) and (2.44) one finds that

$$\frac{\sigma^{0y}(\bar{r})}{\epsilon} = -\frac{\partial}{\partial s} F_z^{0x}(\bar{r}), \quad \bar{r} \in S \tag{2.45}$$

and

$$\frac{\sigma^{0x}(\bar{r})}{\epsilon} = \frac{\partial}{\partial s} F_z^{0y}(\bar{r}), \quad \bar{r} \in S. \tag{2.46}$$

Inserting the relations (2.45) and (2.46) into the expressions (2.39) for the constants C_1, C_2, and C_3 and integrating by parts one finds

$$C_1 = -\int_S F_z^{0y}(\bar{r}) \frac{\partial x}{\partial s} ds, \quad C_2 = \int_S F_z^{0x}(\bar{r}) \frac{\partial y}{\partial s} ds, \quad C_3 = -\int_S F_z^{0x}(\bar{r}) \frac{\partial x}{\partial s} ds \tag{2.47}$$

which gives the three constants in terms of the vector potentials $\bar{F}^{0x}$ and $\bar{F}^{0y}$.

We have now given the exact analytical expressions for the first terms in the TM and TE low-frequency scattered far fields from the cylinder with finite cross section. We found that these expressions were completely determined by calculating three constants that only depended on the shape of the cross section of the cylinder. These three constants are in turn found from the electrostatic solutions for x and y directed impressed electrostatic fields.

Chapter 3

Cylindrical bump on an infinite ground plane

In this chapter the low-frequency expressions are derived for the field diffracted by a cylindrical bump on a ground plane illuminated by a plane wave. Both the bump and the ground plane are perfectly conducting. The bump extends uniformly to infinity in the $+z$ and $-z$ directions and the curve that describes the cross section of the bump in the $x-y$ plane is denoted by B as shown in Figure 3.1. The outward normal to the bump is $\hat{n}$ and the tangent unit vector to B is $\hat{t} = \hat{z} \times \hat{n}$. The equation for the ground plane is $y = 0$ and the intersections between the bump and the ground plane are given by $y = 0, x = \pm d$.

The incident field $(\bar{E}^i, \bar{H}^i)$ and the cylindrical coordinates (r, ϕ, z) are the same as in Section 2. The field reflected in the ground plane $y = 0$ when there is no bump is denoted by $(\bar{E}^r, \bar{H}^r)$. We write the total field as $(\bar{E}, \bar{H}) = (\bar{E}^i, \bar{H}^i) + (\bar{E}^r, \bar{H}^r) + (\bar{E}^d, \bar{H}^d)$, where $(\bar{E}^d, \bar{H}^d)$ is by definition the diffracted field.

It will be shown that the diffracted field is the scattered field in an equivalent scattering problem where the scatterer is the bump plus its image in the ground plane, and the incident field is $(\bar{E}^i + \bar{E}^r, \bar{H}^i + \bar{H}^r)$ as shown in Figure 3.2. The curve that describes the image of the bump in the $y = 0$ plane is denoted by B_i and the cross section of the equivalent scatterer is therefore enclosed by $B \cup B_i$.

By considering the TM and TE polarizations separately it is easily seen that the scattered field in the equivalent scattering problem satisfies the boundary conditions on the ground plane. Since $(\bar{E}^i + \bar{E}^r, \bar{H}^i + \bar{H}^r)$ also satisfies these boundary conditions it is clear from the uniqueness of the solution that the scattered field in the equivalent scattering problem is the diffracted field in the bump scattering problem. This method of images for constructing an equivalent scattering problem was first used by Rayleigh [17] when he solved for scattering from the semi-circular bump.

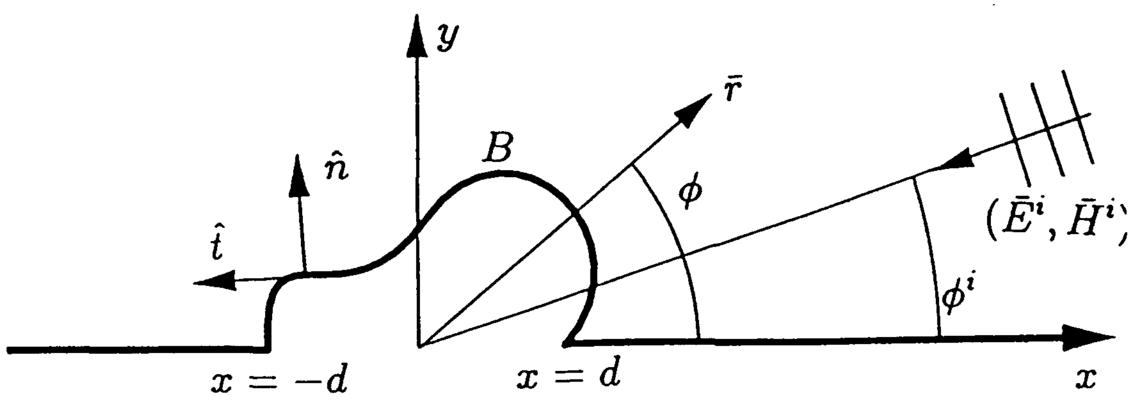

Figure 3.1: Cylindrical bump on a ground plane.

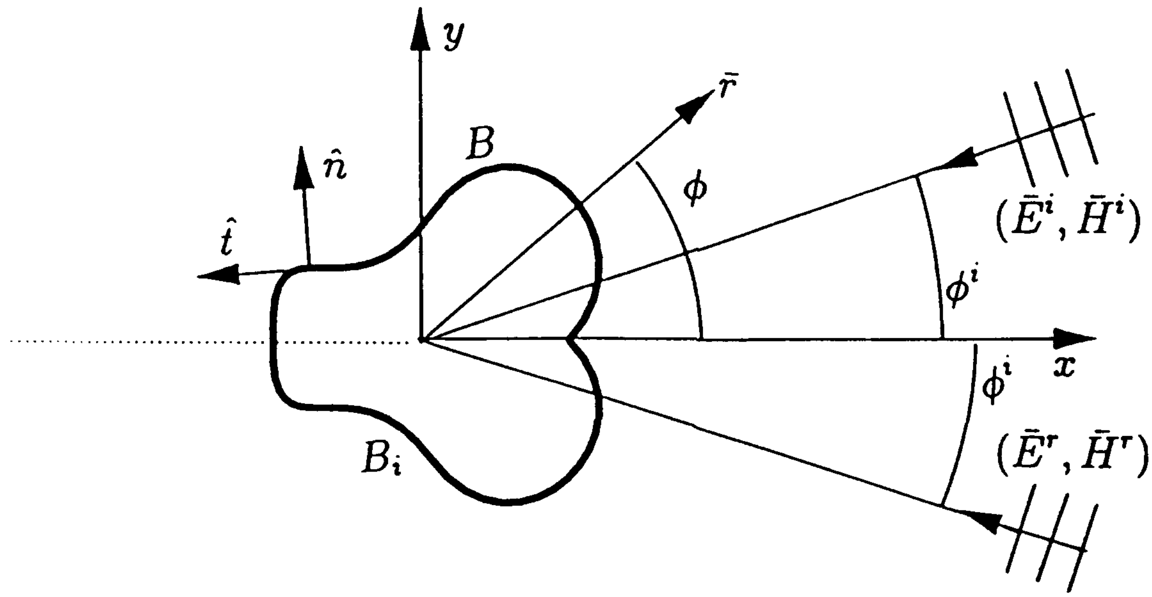

Figure 3.2: Equivalent bump scattering problem.

3.1 TM polarization

We first consider the TM case where the incident electric field is given by (2.1) and the incident magnetic field is given by

$$\bar{H}^i(\bar{r}) = \sqrt{\frac{\epsilon}{\mu}}(-\sin\phi^i\hat{x} + \cos\phi^i\hat{y})e^{jk(x\cos\phi^i + y\sin\phi^i)}. \tag{3.1}$$

The reflected electric field is given by (2.1) with $\hat{z}$ and y replaced by $-\hat{z}$ and $-y$, respectively. Similarly the reflected magnetic field is given by (3.1) with $\hat{y}$ and y replaced by $-\hat{y}$ and $-y$, respectively.

If we use the formula (2.12) for the scattered far field and add the contributions from the incident field $\bar{E}^i$ and the reflected field $\bar{E}^r$ we get zero because the two contributions cancel. Consequently, the diffracted field is zero to order $\frac{1}{\ln kd}$ and we cannot use the results from Section 2.1 because we need higher order terms. We will therefore start by investigating the low-frequency current on the equivalent scatterer given by $B \cup B_i$.

3.1.1 Low-frequency current (TM)

From the integral equation (2.6) it is found that the current K_z on the equivalent scatterer satisfies the integral equation

$$E_z^i(\bar{r}) + E_z^r(\bar{r}) = jk\sqrt{\frac{\mu}{\epsilon}} \fint_{B\cup B_i} G(\bar{r},\bar{r}')K_z(\bar{r}')ds', \quad \bar{r} \in B \cup B_i. \tag{3.2}$$

From the small argument approximation of the Hankel function it is found that

$$G(\bar{r},\bar{r}') = G^0(\bar{r},\bar{r}') - \frac{1}{2\pi}\ln\frac{kd}{2} - \frac{j}{4} - \frac{\gamma}{2\pi} + O((kd)^2\ln kd) \tag{3.3}$$

where $G^0(\bar{r},\bar{r}')$ is the static free-space Green's function in (2.19), γ is Euler's constant, and the last term is non-singular at $\bar{r} = \bar{r}'$, in fact it tends to zero as $\bar{r} \to \bar{r}'$. Because $E_z^i + E_z^r$ is an odd function of y and the equivalent scatterer is symmetric around the $y = 0$ plane it follows that the current K_z is also an odd function of y, that is,

$$K_z(x,y) = -K_z(x,-y). \tag{3.4}$$

Consequently, the total current flowing along the equivalent scatterer is zero, that is,

$$\int_{B\cup B_i} K_z(\bar{r})ds = 0. \tag{3.5}$$

If we expand $E_z^i + E_z^r$ in powers of k, and use the expansion (3.3) of the Green's function along with (3.5) in the integral equation (3.2), we find that the first term K_z^0 in the low-frequency expansion of the current satisfies the integral equation

$$2y\sin\phi^i = \sqrt{\frac{\mu}{\epsilon}} \fint_{B\cup B_i} G^0(\bar{r},\bar{r}')K_z^0(\bar{r}')ds', \quad \bar{r} \in B \cup B_i. \tag{3.6}$$

The uniqueness of the solution to this integral equation has been proven in [15].

We will now show that K_z^0 is the current in the magnetostatic scattering problem with the equivalent scatterer situated in the impressed magnetostatic field

$$\bar{H}^i_{static} = \bar{H}^i|_{k=0} + \bar{H}^r|_{k=0} = -2\sqrt{\frac{\epsilon}{\mu}}\sin\phi^i\hat{x}. \tag{3.7}$$

The magnetostatic current in this scattering problem with the equivalent scatterer is denoted by K_{1z}^0 and a vector potential for the scattered magnetostatic field is given by (2.2) with G replaced by G^0 and $\bar{K}$ replaced by $\hat{z}K_{1z}^0$. From the boundary condition, $\hat{n}\cdot\bar{H} = 0$, for the magnetostatic field we find that the total magnetostatic vector potential must be constant on the conductor, and we therefore require that it be zero there. From the expression (3.7) for the impressed magnetostatic field it is seen that the impressed vector potential must be $-2y\sqrt{\frac{\epsilon}{\mu}}\sin\phi^i + C$ where C is a constant, which is undetermined at this point of the derivation. An integral equation for K_{1z}^0 is thus

$$2y\sqrt{\frac{\epsilon}{\mu}}\sin\phi^i - C = \int_{B\cup B_i} G^0(\bar{r},\bar{r}')K_{1z}^0(\bar{r}')ds', \quad \bar{r}\in B\cup B_i. \tag{3.8}$$

Because the impressed magnetostatic field is parallel to $\hat{x}$ and the equivalent scatterer is symmetric around the plane $y{=}0$ the current excited by this impressed field must be an odd function of y. Therefore, the integral in (3.8) must be an odd function of y and the constant C must be zero.

If we compare (3.8) with $C = 0$ to the integral equation (3.6) for the current $\bar{K}^0$ we see that $K_{1z}^0 = K_z^0$ and we have shown that $\bar{K}^0$ is the current in the magnetostatic problem. The leading term in the low-frequency expansion of the current is thus of order $(kd)^0$. From the integral equation (3.2) it follows that the next term, in general, is a function of kd and $\ln kd$ so that the current cannot be expanded in powers of kd.

It is convenient to define a current K_z^{0B} which is the magnetostatic current when the equivalent scatterer is situated in the impressed magnetostatic field $\hat{x}$. From the expression (3.7) for the impressed magnetostatic field and from the integral equation (3.6) it follows that the current K_z^{0B} is determined by the integral equation

$$-y = \int_{B\cup B_i} G^0(\bar{r},\bar{r}')K_z^{0B}(\bar{r}')ds', \quad \bar{r}\in B\cup B_i \tag{3.9}$$

and the condition (3.4) that it must be an odd function of y. We then can write the low-frequency expansion of the current as

$$K_z = -2\sqrt{\frac{\epsilon}{\mu}}\sin\phi^i K_z^{0B} + K_z^r \tag{3.10}$$

where K_z^{0B} is an odd function of y and satisfies the magnetostatic integral equation (3.9), and $K_z^r \to 0$ as $kd \to 0$.

3.1.2 Low-frequency diffracted far field (TM)

We will now calculate the low-frequency diffracted far field from the bump, which is the scattered far field in the equivalent scattering problem of the bump and its image.

Using the electric field expression (2.5) and the far-field approximation (2.10) to the Green's function we find that the diffracted far field is

$$E_z^d(\bar{r}) \sim -jk\frac{e^{-j\pi/4}}{2\sqrt{2\pi}}\frac{e^{-jkr}}{\sqrt{kr}}\sqrt{\frac{\mu}{\epsilon}}\int_{B\cup B_i}[1+jk(x'\cos\phi+y'\sin\phi)+O((kd)^2)]K_z(\bar{r}')ds'. \quad (3.11)$$

Because the current is an odd function of y and because the equivalent scatterer is symmetric around the ground plane $y=0$ we find that

$$\int_{B\cup B_i} xK_z(\bar{r})ds = 0, \quad \int_{B\cup B_i} yK_z(\bar{r})ds = 2\int_B yK_z(\bar{r})ds. \quad (3.12)$$

Furthermore, using the fact that the total current is zero (3.5) together with the expansion (3.10) of the current we find that the diffracted far field (3.11) to order $(kd)^2$ is given by

$$E_z^d(\bar{r}) \sim -(kd)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{e^{-jkr}}{\sqrt{kr}}\frac{1}{d^2}B_0\sin\phi\,\sin\phi^i \quad (3.13)$$

where

$$B_0 = \int_B yK_z^{0B}(\bar{r})ds. \quad (3.14)$$

The magnetostatic current K_z^{0B} is an odd function of y and satisfies the static integral equation (3.9).

From the formula (3.13) it is seen that the diffracted field is the field of a magnetic dipole in the x-direction [11, sec.3.8].

3.2 TE polarization

In the TE case the incident magnetic and electric fields are given in (2.13) and (2.14). Because the contributions from the incident and reflected fields in the TE case do not cancel we can use the results from Section 2.2 directly so we do not have to investigate the current on the equivalent scatterer.

3.2.1 Low-frequency diffracted far field (TE)

If we let A_B denote the area of the region bounded by B and the line $y=0, |x|\le d$ (see Figure 3.1) then the diffracted far field is given by

$$H_z^d(\bar{r}) = H_z^s(\bar{r})|_{A_S=2A_B} + H_z^s(\bar{r})|_{A_S=2A_B,\phi^i\to-\phi^i} \quad (3.15)$$

where H_z^s is the scattered field given in (2.38). The first term in (3.15) is the contribution from $(\bar{E}^i, \bar{H}^i)$ and the second term is the contribution from $(\bar{E}^r, \bar{H}^r)$. Because of the symmetry of the scatterer we find that

$$\int_{B\cup B_i} x\sigma^{0y}(\bar{r})ds = 0, \quad \int_{B\cup B_i} y\sigma^{0y}(\bar{r})ds = 2\int_B y\sigma^{0y}(\bar{r})ds \tag{3.16}$$

where σ^{0x} and σ^{0y} are the electrostatic charges in the electrostatic problems with the equivalent scatterer situated in the impressed electrostatic fields $\hat{x}$ and $\hat{y}$, respectively.

With substitution of (3.16) into the field expression (2.38), equation (3.15) becomes

$$H_z^d(\bar{r}) \sim -(kd)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{e^{-jkr}}{\sqrt{kr}}\frac{1}{d^2}[A_B + \cos\phi\cos\phi^i \int_B \frac{\sigma^{0y}(\bar{r}')}{\epsilon}y'ds']. \tag{3.17}$$

The diffracted field therefore consists of the contribution from a magnetic dipole in the $z-$direction and an electric dipole in the y-direction [11, sec.3.8].

Alternatively, using (2.45) the integral in (3.17) may be written in terms of the vector potential $\bar{F}^{0x}$ satisfying (2.42) with S replaced by $B \cup B_i$.

We will now show that, remarkably, the integral in the expression (3.17) for the TE diffracted field equals the negative of the constant B_0 in (3.14) occurring in the expression (3.13) for the TM diffracted field. To do this we introduce a scalar potential for the electrostatic field and use the fact that the impressed scalar potential must be $C - y$ where C is a constant. Furthermore the boundary condition of zero tangential electric field requires that the total scalar potential be constant on the equivalent scatterer, and we find that the charge σ^{0y} therefore must satisfy the integral equation

$$y - C = \int_{B\cup B_i} G^0(\bar{r},\bar{r}')\frac{\sigma^{0y}(\bar{r}')}{\epsilon}ds', \quad \bar{r} \in B \cup B_i \tag{3.18}$$

where C is undetermined at this point of the derivation.

Because the impressed electrostatic field is symmetric around the ground plane $y = 0$ the electrostatic charge σ^{0y} must be an odd function of y. Consequently, the integral in (3.18) must be an odd function of y and the constant C must be zero.

Comparing the integral equation (3.18) with $C = 0$ to the integral equation (3.9) for the current K_z^{0B} one finds that $\epsilon K_z^{0B} = -\sigma^{0y}$ and therefore the TE diffracted field is given by

$$H_z^d(\bar{r}) \sim -(kd)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{e^{-jkr}}{\sqrt{kr}}\frac{1}{d^2}[A_B - B_0\cos\phi\,\cos\phi^i] \tag{3.19}$$

where B_0 is given in (3.14) or (3.17).

We have now derived the exact expressions for the leading terms in the low-frequency expansions of the diffracted far fields from arbitrarily shaped bumps. Our general analysis has proven the rather remarkable result that the constant in the TM magnetic dipole term equals the negative of the constant in the TE electric dipole term for arbitrarily shaped bumps. In the following chapter we will show that this result also holds for diffraction from a two-dimensional dent in a ground plane.

Chapter 4

Cylindrical dent in an infinite ground plane

In this chapter we derive the low-frequency expressions for the fields diffracted by a cylindrical dent in a ground plane illuminated by a plane wave. Both the dent and the ground plane are perfectly conducting. The dent extends uniformly to infinity in the $+z$ and $-z$ directions and the curve that describes the cross section of the dent in the $x-y$ plane is denoted by D as shown in Figure 4.1. The ground plane is given by $y = 0$ and the dent intersects the ground plane at $y = 0, x = \pm d$. The line segment given by $y = 0, |x| \leq d$ that caps the dent is called the aperture and is denoted by A. The image of D with respect to the ground plane $y = 0$ is denoted by D_i, and the part of the upper half plane ($y > 0$) which is outside the image D_i is called F. The normal to D is $\hat{n}$ and the tangent unit vector to D is $\hat{t} = \hat{z} \times \hat{n}$. Normal and tangent unit vectors to A are $\hat{y}$ and $-\hat{x}$, respectively. The incident field $(\bar{E}^i, \bar{H}^i)$, the reflected field $(\bar{E}^r, \bar{H}^r)$, the diffracted field $(\bar{E}^d, \bar{H}^d)$, and the cylindrical coordinates (r, ϕ, z) are defined as in the previous chapter.

Although the dent problem is very similar to the bump problem, the analysis of the dent is much more complicated and thus warrants a separate analysis. The main reason for the complication is that the dent problem, unlike the bump problem, cannot be reduced using image theory to an equivalent problem that involves only a cylinder of finite cross section without a ground plane. One therefore has to deal with integral equations more complicated than the usual two-dimensional electric and magnetic field integral equations. It is essential to the derivation that the integral equation involves only fields and currents in a region whose maximum dimension tends to zero as the characteristic dimension of the dent tends to zero. We found that the most convenient integral equations for this purpose were coupled integral equations first derived by Asvestas and Kleinman [18]. In Hansen and Yaghjian [19] these integral equations have been derived straightforwardly by means of the Stratton-Chu formulas [20]. Here they will be derived as in [18] by means of Green's identities.

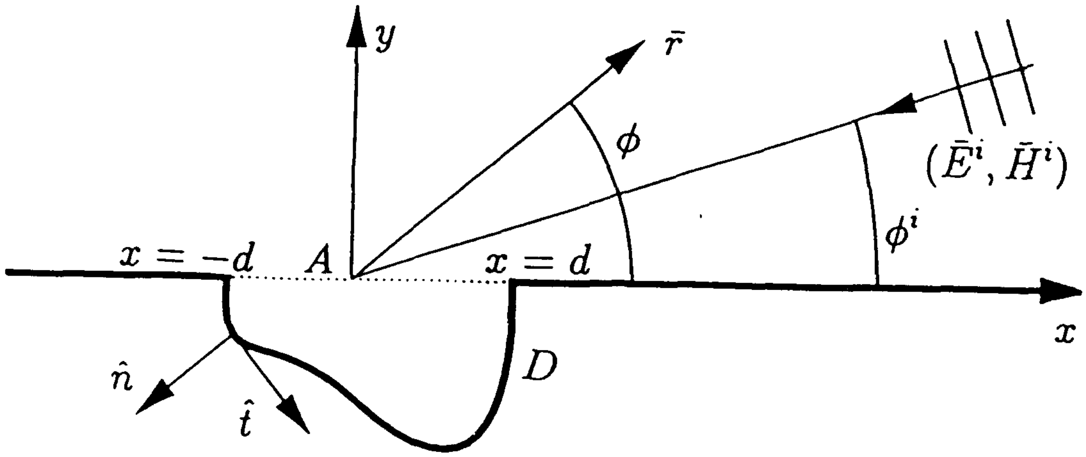

Figure 4.1: Cylindrical dent in a ground plane.

4.1 Transverse magnetic (TM) polarization

The incident electric and magnetic fields are here given by

$$\bar{E}^i(\bar{r}) = \hat{z}e^{jk(x\cos\phi^i + y\sin\phi^i)} \tag{4.1}$$

and

$$\bar{H}^i(\bar{r}) = \sqrt{\frac{\epsilon}{\mu}}(-\sin\phi^i\hat{x} + \cos\phi^i\hat{y})e^{jk(x\cos\phi^i + y\sin\phi^i)}. \tag{4.2}$$

The reflected electric field is given by (4.1) with $\hat{z}$ and y replaced by $-\hat{z}$ and $-y$, respectively. Similarly, the reflected magnetic field is given by (4.2) with $\hat{y}$ and y replaced by $-\hat{y}$ and $-y$, respectively.

4.1.1 Integral equation for the current (TM)

We will start by giving a short derivation of Asvestas and Kleinman's [18] coupled TM integral equations for the current in the dent and the magnetic field in the aperture. We need the details of this derivation to derive our low-frequency results.

Let $S_1(R)$ be the part of the x-axis given by $|x| < R$ with $R > d$ and let $S_2(R)$ be the part of the circle $r = R$ which is in the region $y \geq 0$. Furthermore, let $\hat{n}_S$ be that normal unit vector to $S_1(R) \cup S_2(R)$ which points into the region enclosed by $S_1(R) \cup S_2(R)$ shown in Figure 4.2. Let G_D be the Dirichlet Green's function for the region $y \geq 0$, that is,

$$G_D(\bar{r}, \bar{r}') = G(\bar{r}, \bar{r}') - G(\bar{r}, \bar{r}_i') \tag{4.3}$$

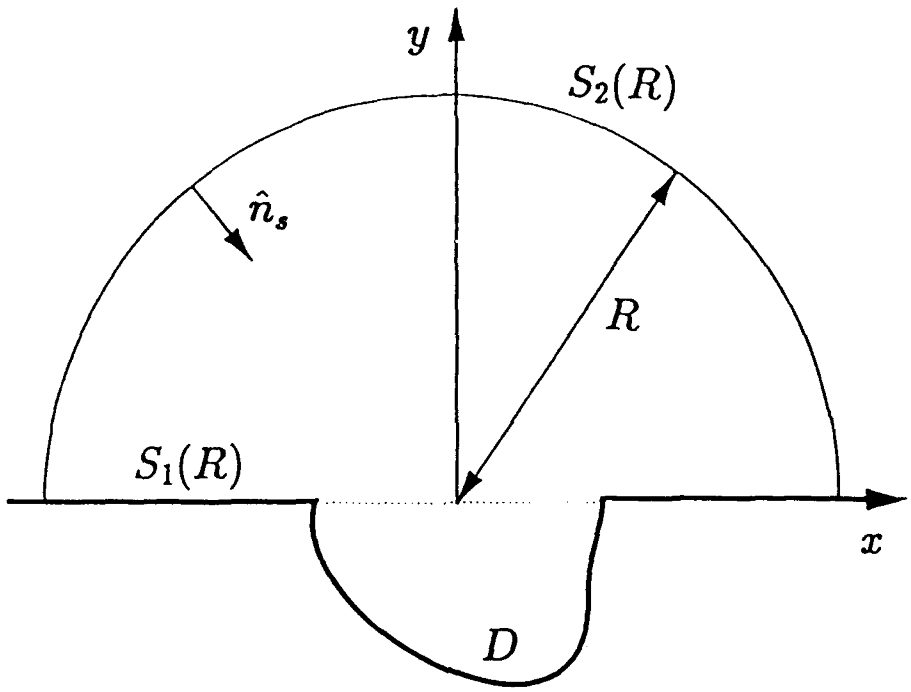

Figure 4.2: Half-circle in the upper half-plane $y > 0$.

where $G(\bar{r}, \bar{r}')$ is the two-dimensional free-space Green's function in (2.4) and

$$\bar{r}_i = x\hat{x} - y\hat{y} \tag{4.4}$$

is the image in the ground plane of the point $\bar{r}$. If we use Green's second identity and the fact that the diffracted field E_z^d satisfies the scalar homogeneous Helmholtz equation we find that the diffracted field satisfies

$$E_z^d(\bar{r}) = -\int_{S_1(R)\cup S_2(R)} [G_D(\bar{r}, \bar{r}')\nabla' E_z^d(\bar{r}') - E_z^d(\bar{r}')\nabla' G_D(\bar{r}, \bar{r}')] \cdot \hat{n}'_S ds', \quad y > 0. \tag{4.5}$$

Because of the boundary conditions for G_D and E_z^d on the ground plane, (4.5) becomes

$$\begin{aligned} E_z^d(\bar{r}) &= \int_A E_z^d(\bar{r}') \frac{\partial}{\partial y'} G_D(\bar{r}, \bar{r}') ds' \\ &+ \int_{S_2(R)} [E_z^d(\bar{r}') \frac{\partial}{\partial r'} G_D(\bar{r}, \bar{r}') - G_D(\bar{r}, \bar{r}') \frac{\partial}{\partial r'} E_z^d(\bar{r}')] ds', \quad y > 0. \end{aligned} \tag{4.6}$$

Using the asymptotic relations for G_D and that E_z^d satisfies the two-dimensional radiation condition [11, sec.1.34] one may show that the last integral in (4.6) tends to zero as $R \to \infty$. Thus (4.6) implies

$$E_z^d(\bar{r}) = \int_A E_z^d(\bar{r}') \frac{\partial}{\partial y'} G_D(\bar{r}, \bar{r}') ds', \quad y > 0. \tag{4.7}$$

Using the property of the Dirichlet's Green's function that $\frac{\partial}{\partial y'}G_D(\bar{r},\bar{r}') = 2\frac{\partial}{\partial y'}G(\bar{r},\bar{r}'),\ \bar{r}' \in A$ and that $E_z^i(\bar{r}) + E_z^r(\bar{r}) = 0,\ \ \bar{r} \in A$, (4.7) becomes

$$E_z(\bar{r}) = 2\int_A E_z(\bar{r}')\frac{\partial}{\partial y'}G(\bar{r},\bar{r}')ds' + E_z^i(\bar{r}) + E_z^r(\bar{r}),\ \ y > 0 \tag{4.8}$$

which is an expression for the total field outside the dent in terms of the field in the aperture.

Taking the curl of (4.8) to get the magnetic field, we find

$$\begin{aligned} -jk\sqrt{\frac{\mu}{\epsilon}}H_x(\bar{r}) = & - 2\int_A E_z(\bar{r}')\frac{\partial^2}{\partial y'^2}G(\bar{r},\bar{r}')ds' \\ & - jk\sqrt{\frac{\mu}{\epsilon}}[H_x^i(\bar{r}) + H_x^r(\bar{r})],\ \ y > 0. \end{aligned} \tag{4.9}$$

Because the integral in this equation is an even function of y we have

$$\begin{aligned} -jk\sqrt{\frac{\mu}{\epsilon}}H_x(\bar{r}_i) = & - 2\int_A E_z(\bar{r}')\frac{\partial^2}{\partial y'^2}G(\bar{r},\bar{r}')ds' \\ & - jk\sqrt{\frac{\mu}{\epsilon}}[H_x^i(\bar{r}_i) + H_x^r(\bar{r}_i)],\ \ \bar{r} \in int(A \cup D) \end{aligned} \tag{4.10}$$

where $\bar{r}_i$ is given in (4.4) and $int(A \cup D)$ is the region bounded by $A \cup D$. To get rid of the integral involving the second derivative of the Green's function we will derive another relation involving this second derivative and then combine these two relations.

If we recall that E_z satisfies the homogeneous Helmholtz equation and apply Green's second identity in the region bounded by $A \cup D$ we find that

$$E_z(\bar{r}) = \int_{A\cup D}[G(\bar{r},\bar{r}')\frac{\partial}{\partial n'}E_z(\bar{r}') - E_z(\bar{r}')\frac{\partial}{\partial n'}G(\bar{r},\bar{r}')]ds',\ \ \bar{r} \in int(A \cup D). \tag{4.11}$$

Taking the curl of (4.11)and applying the boundary condition of zero tangential electric field in the dent produces the following equation for the $x-$component of the magnetic field,

$$\begin{aligned} -jk\sqrt{\frac{\mu}{\epsilon}}H_x(\bar{r}) = & - \int_{A\cup D}\frac{\partial}{\partial y'}G(\bar{r},\bar{r}')\frac{\partial}{\partial n'}E_z(\bar{r}')ds' \\ & + \int_A E_z(\bar{r}')\frac{\partial^2}{\partial y'^2}G(\bar{r},\bar{r}')ds',\ \ \bar{r} \in int(A \cup D). \end{aligned} \tag{4.12}$$

Insert the integral in (4.10) involving the second derivative of the Green's function into (4.12) to get

$$\begin{aligned} \frac{j}{k}\sqrt{\frac{\epsilon}{\mu}}\int_{A\cup D}\frac{\partial}{\partial y'}G(\bar{r},\bar{r}')\frac{\partial}{\partial n'}E_z(\bar{r}')ds' = -H_x(\bar{r}) + \frac{1}{2}[-H_x(\bar{r}_i) \\ +H_x^i(\bar{r}_i) + H_x^r(\bar{r}_i)],\ \ \bar{r} \in int(A \cup D). \end{aligned} \tag{4.13}$$

We now let $\bar{r}$ approach the aperture and use the results of the integration near the singularity in [14, App.], the induction law of Maxwell's equations, the following expression for the current on D,

$$K_z(\bar{r}) = \frac{j}{k}\sqrt{\frac{\epsilon}{\mu}}\frac{\partial}{\partial n}E_z(\bar{r}), \quad \bar{r} \in D, \tag{4.14}$$

and the relation

$$\frac{\partial}{\partial y'}G(\bar{r},\bar{r}') = 0, \quad y = y' = 0, \quad x \neq x' \tag{4.15}$$

to get from (4.13)

$$\int_D \frac{\partial}{\partial y'}G(\bar{r},\bar{r}')K_z(\bar{r}')ds' = H_x^i(\bar{r}) - H_x(\bar{r}), \quad \bar{r} \in A \tag{4.16}$$

which is a relation between the current in the dent and the magnetic field in the aperture.

To determine the current in the dent we need one more relation of this kind. To obtain this we apply Green's second identity in the region bounded by $A \cup D$ and find that

$$\begin{aligned} E_z(\bar{r}) &= \int_D G_N(\bar{r},\bar{r}')\frac{\partial}{\partial n'}E_z(\bar{r}')ds' \\ &+ 2\int_A G(\bar{r},\bar{r}')\frac{\partial}{\partial y'}E_z(\bar{r}')ds', \quad \bar{r} \in int(A \cup D) \end{aligned} \tag{4.17}$$

where G_N is the Neumann Green's function for the region $y \geq 0$, that is,

$$G_N(\bar{r},\bar{r}') = G(\bar{r},\bar{r}') + G(\bar{r},\bar{r}_i'). \tag{4.18}$$

Let $\bar{r}$ approach D in (4.17), and use the boundary condition of zero tangential electric field in the dent along with the expression (4.14) for the current, and the induction law to get the second integral equation

$$\int_D G_N(\bar{r},\bar{r}')K_z(\bar{r}')ds' + 2\int_A G(\bar{r},\bar{r}')H_x(\bar{r}')ds' = 0, \quad \bar{r} \in D. \tag{4.19}$$

The relations (4.16) and (4.19) constitute a pair of coupled integral equations that determine the current K_z in D and the magnetic field H_x in A.

4.1.2 Low-frequency current (TM)

We will now use the coupled integral equations (4.16) and (4.19) to determine the first term in the low-frequency expansion of the current in the dent.

Using the low-frequency expansion (2.18) of the derivative of the Green's function in the integral equation (4.16) we find that

$$\int_D \frac{\partial}{\partial y'}G^0(\bar{r},\bar{r}')K_z^0(\bar{r}')ds' = H_x^i(\bar{r})|_{k=0} - H_x^0(\bar{r}), \quad \bar{r} \in A \tag{4.20}$$

where K_z^0 and H_x^0 are the first terms in the low-frequency expansions of the current in the dent and the magnetic field in the aperture, respectively. G^0 is the static two-dimensional free-space Green's function given in (2.19). By using the low-frequency expansion of the free-space Green's function (3.3) and the corresponding expansion of the Neumann Green's function in the integral equation (4.19) we find that

$$\int_D K_z^0(\bar{r}')ds' + \int_A H_x^0(\bar{r}')ds' = 0 \tag{4.21}$$

and

$$\fint_D G_N^0(\bar{r},\bar{r}')K_z^0(\bar{r}')ds' + 2\int_A G^0(\bar{r},\bar{r}')H_x^0(\bar{r}')ds' = 0, \quad \bar{r} \in D. \tag{4.22}$$

The relations (4.20), (4.21), and (4.22) constitute a set of coupled integral equations that determine the low-frequency current on D. In [15] it is shown that these three coupled integral equations have a unique solution.

We will now show that these integral equations are the integral equations for the magnetostatic current when the scatterer is situated in the impressed magnetostatic field

$$\bar{H}^i_{static} = -\sqrt{\frac{\epsilon}{\mu}}\sin\phi^i\hat{x}. \tag{4.23}$$

We consider the magnetostatic problem in which the magnetostatic solution $\bar{H}_1^0$ satisfies $\hat{n}\cdot\bar{H}_1^0 = 0$ and $\hat{n}\times\bar{H}_1^0 = -\bar{K}_1^0$ on the conductor. Furthermore, it can be proven that these boundary conditions make the total static field zero below the ground plane and dent. Since the total magnetostatic field is zero below the ground plane and dent it is convenient to divide the total field into three components as $\bar{H}_1^0 = \bar{H}^i_{static} + \bar{H}^r_{static} + \bar{H}^d_{static}$ where $\bar{H}^r_{static}$ is the scattered field when there is no dent and $\bar{H}^d_{static}$ is by definition the diffracted magnetostatic field.

We introduce a vector potential $\bar{A}_1^0 = A_{1z}^0\hat{z}$ such that $\bar{H}_1^0 = \nabla\times\bar{A}_1^0$, $\nabla^2 A_{1z}^0 = 0$, and $A_{1z}^0 = 0$ on the conductor and see that A_{1z}^0 satisfies Laplace's equation and the Dirichlet boundary condition on the conductor just as the $z-$component of an electrostatic field. Therefore the potential for the diffracted magnetostatic field A_{1z}^{0d} satisfies (4.6) with G_D replaced by G_D^0 which is the static Dirichlet Green's function for the region $y \geq 0$, that is,

$$\begin{aligned} A_{1z}^{0d}(\bar{r}) &= \int_A A_{1z}^{0d}(\bar{r}')\frac{\partial}{\partial y'}G_D^0(\bar{r},\bar{r}')ds' \\ &+ \int_{S_2(R)}[A_{1z}^{0d}(\bar{r}')\frac{\partial}{\partial r'}G_D^0(\bar{r},\bar{r}') - G_D^0(\bar{r},\bar{r}')\frac{\partial}{\partial r'}A_{1z}^{0d}(\bar{r}')]ds', \quad y > 0. \end{aligned} \tag{4.24}$$

The diffracted magnetostatic field, at infinity, behaves as if it were scattered from a cylinder with a finite cross section. This assumption is confirmed by the fact that the corresponding magnetostatic field for the bump (see Chapter 3) indeed behaved as if it was scattered by a cylinder with finite cross section. From [20, sec. 4.4] it then follows that in the two-dimensional case $A_{1z}^{0d} \sim |\nabla \ln r| = \frac{1}{r}$ and $\frac{\partial}{\partial r}A_{1z}^{0d} \sim= \frac{1}{r^2}$ as $r \to \infty$. Furthermore, since

$G_D^0(r,r') \sim \ln r$ and $\frac{\partial}{\partial r'} G_D^0(\bar{r},\bar{r}') \sim \frac{1}{r}$ as $r \to \infty$ we find that the last integral in (4.24) tends to zero as $R \to \infty$. Hence,

$$A_{1z}^{0d}(\bar{r}) = \int_A A_{1z}^{0d}(\bar{r}') \frac{\partial}{\partial y'} G_D^0(\bar{r},\bar{r}') ds', \ \ y > 0 \tag{4.25}$$

which is the static version of (4.7). We now use exactly the same procedure which in the time-harmonic case led from (4.7) to (4.16) and find that (4.25) leads to the relation

$$\int_D \frac{\partial}{\partial y'} G^0(\bar{r},\bar{r}') K_{1z}^0(\bar{r}') ds' = H_{static\, x}^i(\bar{r}) - H_{1x}^0(\bar{r}), \ \ \bar{r} \in A \tag{4.26}$$

which corresponds to (4.16). Similarly one finds

$$\fint_D G_N^0(\bar{r},\bar{r}') K_{1z}^0(\bar{r}') ds' + 2\int_A G^0(\bar{r},\bar{r}') H_{1x}^0(\bar{r}') ds' = 0, \ \ \bar{r} \in D \tag{4.27}$$

which corresponds to (4.19). Because the z-component of the vector potential $\bar{A}_1^0$ satisfies Laplace's equation the field $\bar{H}_1^0$ and the current $\bar{K}_1^0$ also satisfy the relation (4.21). We have now shown that the magnetostatic current and field satisfy the integral equations (4.20), (4.21), and (4.22) and consequently the first term in the low-frequency expansion of the current in the dent equals this magnetostatic current.

If we let $\bar{K}^{0D}$ and $\bar{H}^{0D}$ be the magnetostatic current and field in the magnetostatic field problem where the scatterer is situated in the impressed magnetostatic field $\frac{1}{2}\hat{x}$ we obtain from (4.23) the following low-frequency expansion of the current in the dent

$$K_z = -2\sqrt{\frac{\epsilon}{\mu}} \sin\phi^i K_z^{0D} + K_z^r \tag{4.28}$$

where $K_z^r \to 0$ as $kd \to 0$. From the integral equations (4.20), (4.21), and (4.22) it follows that $\bar{K}^{0D}$ and $\bar{H}^{0D}$ satisfy the static integral equations

$$\int_D \frac{\partial}{\partial y'} G^0(\bar{r},\bar{r}') K_z^{0D}(\bar{r}') ds' = \frac{1}{2} - H_x^{0D}(\bar{r}), \ \ \bar{r} \in A \tag{4.29}$$

$$\int_D K_z^{0D}(\bar{r}') ds' + \int_A H_x^{0D}(\bar{r}') ds' = 0 \tag{4.30}$$

and

$$\fint_D G_N^0(\bar{r},\bar{r}') K_z^{0D}(\bar{r}') ds' + 2\int_A G^0(\bar{r},\bar{r}') H_x^{0D}(\bar{r}') ds' = 0, \ \ \bar{r} \in D. \tag{4.31}$$

(In [19] (4.29), (4.30), and (4.31) are derived directly from the Stratton-Chu formulas.) In summary, we have determined the first term in the low-frequency expansion of the current in the dent and in the next section we integrate this low-frequency current to get the low-frequency diffracted far field.

4.1.3 Low-frequency diffracted far field (TM)

To determine an expression for the low-frequency diffracted far field we will first derive a relation between the current in the dent and the diffracted field in F, the region in the upper half plane ($y > 0$) outside the image of the dent.

When $\bar{r} \in F$ and $\bar{r}' \in int(A \cup D)$ the Dirichlet Green's function $G_D(\bar{r}, \bar{r}')$ satisfies the homogeneous Helmholtz equation, and Green's second identity gives

$$0 = \int_{A \cup D} [G_D(\bar{r}, \bar{r}') \frac{\partial}{\partial n'} E_z(\bar{r}') - E_z(\bar{r}') \frac{\partial}{\partial n'} G_D(\bar{r}, \bar{r}')] ds', \quad \bar{r} \in F. \tag{4.32}$$

With the boundary conditions $E_z = 0$ on D and $G_D = 0$ on A (4.32) becomes

$$\int_D G_D(\bar{r}, \bar{r}') \frac{\partial}{\partial n'} E_z(\bar{r}') ds' = \int_A E_z(\bar{r}') \frac{\partial}{\partial y'} G_D(\bar{r}, \bar{r}') ds', \quad \bar{r} \in F. \tag{4.33}$$

Use of the expression (4.7) for the diffracted field, the fact that $E_z^i + E_z^r = 0$ on A, and the relation (4.14) between the current and the electric field transform (4.33) to

$$E_z^d(\bar{r}) = -jk\sqrt{\frac{\mu}{\epsilon}} \int_D G_D(\bar{r}, \bar{r}') K_z(\bar{r}') ds', \quad \bar{r} \in F \tag{4.34}$$

which is the relation between the current in the dent and the diffracted field in F. This expression was first derived by Asvestas and Kleinman [18].

From the asymptotic expansion of the Hankel function one finds that

$$G_D(\bar{r}, \bar{r}') \sim \frac{e^{j\pi/4}}{\sqrt{2\pi}} \frac{e^{-jkr}}{\sqrt{kr}} k \sin\phi [y' + O(kd)], \quad r \to \infty. \tag{4.35}$$

The expansion (4.35) and the current expansion (4.28) inserted into the field expression (4.34) gives the diffracted far field to order $(kd)^2$ as

$$E_z^d(\bar{r}) \sim -(kd)^2 e^{-j\pi/4} \sqrt{\frac{2}{\pi}} \frac{e^{-jkr}}{\sqrt{kr}} \frac{1}{d^2} D_0 \sin\phi \sin\phi^i \tag{4.36}$$

where

$$D_0 = \int_D K_z^{0D}(\bar{r}') y' ds' \tag{4.37}$$

which is the final expression for the TM low-frequency diffracted far field of the dent D in the ground plane. We see that this TM low-frequency far field is that of a magnetic dipole in the x-direction [11, sec.3.8].

4.2 Transverse electric (TE) polarization

The incident electric and magnetic fields are here given by

$$\bar{H}^i(\bar{r}) = \hat{z} e^{jk(x\cos\phi^i + y\sin\phi^i)} \tag{4.38}$$

and

$$\bar{E}^i(\bar{r}) = \sqrt{\frac{\epsilon}{\mu}}(\sin\phi^i\hat{x} - \cos\phi^i\hat{y})e^{jk(x\cos\phi^i + y\sin\phi^i)}. \tag{4.39}$$

The reflected magnetic field is given by (4.38) with y replaced by $-y$. Similarly, the reflected electric field is given by (4.39) with $\hat{x}$ and y replaced by $-\hat{x}$ and $-y$, respectively.

4.2.1 Integral equation for the current (TE)

We will start by giving a short derivation of Asvestas and Kleinman's [18] coupled TE integral equations for the current in the dent and the magnetic field in the aperture since we need the details of the derivation to derive our low-frequency results.

Performing a derivation similar to the one that led to (4.8) yields the following relation between the magnetic field in the region $y > 0$ and the magnetic field in the aperture.

$$H_z(\bar{r}) = H_z^i(\bar{r}) + H_z^r(\bar{r}) - 2\int_A G(\bar{r},\bar{r}')\frac{\partial}{\partial y'}H_z(\bar{r}')ds', \quad y > 0. \tag{4.40}$$

Since the integral in this equation is an even function of y we have

$$H_z(\bar{r}_i) = H_z^i(\bar{r}_i) + H_z^r(\bar{r}_i) - 2\int_A G(\bar{r},\bar{r}')\frac{\partial}{\partial y'}H_z(\bar{r}')ds', \quad \bar{r} \in int(A \cup D) \tag{4.41}$$

where $\bar{r}_i$ is the image in the ground plane of the point $\bar{r}$ given in (4.4). To get rid of the integral involving the derivative of the magnetic field in the aperture we will derive one more relation containing this integral.

Using Green's second identity and the fact that H_z satisfies the homogeneous Helmholtz equation we obtain

$$\begin{aligned} H_z(\bar{r}) &= \int_A G(\bar{r},\bar{r}')\frac{\partial}{\partial y'}H_z(\bar{r}')ds' \\ &- \int_{A\cup D} H_z(\bar{r}')\frac{\partial}{\partial n'}G(\bar{r},\bar{r}')ds', \quad \bar{r} \in int(A \cup D). \end{aligned} \tag{4.42}$$

Substitution of the integral in (4.41) into (4.42) yields

$$\begin{aligned} \int_{A\cup D} H_z(\bar{r}')\frac{\partial}{\partial n'}G(\bar{r},\bar{r}')ds' = -H_z(\bar{r}) + \frac{1}{2}[-H_z(\bar{r}_i) \\ +H_z^i(\bar{r}_i) + H_z^r(\bar{r}_i)], \quad \bar{r} \in int(A \cup D). \end{aligned} \tag{4.43}$$

Letting $\bar{r}$ approach A, using the result of the integration near the singularity from [14, App.] with the boundary relations

$$\frac{\partial}{\partial y'}G(\bar{r},\bar{r}') = 0, \quad y = y' = 0, \quad x \neq x', \tag{4.44}$$

and

$$H_z(\bar{r}) = K_t(\bar{r}), \quad \bar{r} \in D \tag{4.45}$$

(4.43) converts to

$$\int_D K_t(\bar{r}')\frac{\partial}{\partial n'}G(\bar{r},\bar{r}')ds' = -H_z(\bar{r}) + H_z^i(\bar{r}), \quad \bar{r} \in A. \tag{4.46}$$

This is a relation between the current in the dent and the magnetic field in the aperture. To determine the current in the dent we need one more relation of this kind. Using Green's second identity, the fact that H_z satisfies the scalar Helmholtz equation, and the boundary condition that $\frac{\partial}{\partial n}H_z = 0$ on D yield that

$$H_z(\bar{r}) = -\int_{A\cup D} H_z(\bar{r})\frac{\partial}{\partial n'}G_D(\bar{r},\bar{r}')ds', \quad \bar{r} \in int(A \cup D). \tag{4.47}$$

Letting $\bar{r}$ approach D in (4.47), using the result of the integration near the singularity from [14, App.], and using (4.45) gives us

$$\frac{1}{2}K_t(\bar{r}) = -\fint_D K_t(\bar{r}')\frac{\partial}{\partial n'}G_D(\bar{r},\bar{r}')ds' - 2\int_A H_z(\bar{r}')\frac{\partial}{\partial y'}G(\bar{r},\bar{r}')ds', \quad \bar{r} \in D \tag{4.48}$$

which is the second relation between the current in the dent and the magnetic field in the aperture. The relations (4.46) and (4.48) constitute a pair of coupled integral equations for the current in the dent. (They also can be derived directly from the Stratton-Chu formulas [19].)

4.2.2 Low-frequency current (TE)

We will now use the coupled integral equations (4.46) and (4.48) to determine the first two terms in the low-frequency expansion of the current in the dent. Using the low-frequency expansion (2.18) for $\frac{\partial}{\partial n}G$ and the similar expansion for $\frac{\partial}{\partial n}G_N$ in the integral equations (4.46) and (4.48) we find that

$$\int_D K_t^0(\bar{r}')\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')ds' = -H_z^0(\bar{r}) + 1, \quad \bar{r} \in A \tag{4.49}$$

and

$$\frac{1}{2}K_t^0(\bar{r}) = -\fint_D K_t^0(\bar{r}')\frac{\partial}{\partial n'}G_D^0(\bar{r},\bar{r}')ds' - 2\int_A H_z^0(\bar{r}')\frac{\partial}{\partial y'}G^0(\bar{r},\bar{r}')ds', \quad \bar{r} \in D \tag{4.50}$$

where K_t^0 and H_z^0 are the first terms in the low-frequency expansions of the current K_t and magnetic field H_z, respectively. In [15] it is shown that these two coupled integral equations have a unique solution.

We will now show that these are the integral equations for the magnetostatic current and field in the case where the scatterer is situated in the impressed magnetostatic field

$$\bar{H}^i_{static} = \bar{H}^i|_{k=0} = \hat{z}. \tag{4.51}$$

To do this we define a reflected and a diffracted magnetostatic field as in the TM case (see the discussion after (4.23)). We find that the reflected magnetostatic field is $\hat{z}$ for $y > 0$ and $-\hat{z}$ for $y < 0$. Similarly, it is found that the diffracted magnetostatic field is $2\hat{z}$ in the region bounded by $A \cup D$ and zero everywhere else. These observations make it very easy to use the same procedure that gave us the time-harmonic equations (4.46) and (4.48) to show that (4.49) and (4.50) indeed are the equations for this magnetostatic field which simply equals $2\hat{z}$ in the free-space region. Thus the current in the dent and the magnetic field in the aperture may be expanded as

$$K_t = 2 + K_t^r, \quad H_z = 2 + H_z^r \tag{4.52}$$

where K_t^r and H_z^r tends to zero as $kd \to 0$. We have hereby determined the first term in the low-frequency expansion of the current in the dent and we will now determine the second term in this low-frequency expansion.

If we insert the expansions (4.52) of the current and field as well as the low-frequency expansions of the Green's functions into the integral equations (4.46) and (4.48) we get from the static integral equations (4.49) and (4.50)

$$\int_D K_t^1(\bar{r}')\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')ds' = -H_z^1(\bar{r}) + jkx\cos\phi^i, \quad \bar{r} \in A \tag{4.53}$$

and

$$\frac{1}{2}K_t^1(\bar{r}) = -\int_D K_t^1(\bar{r}')\frac{\partial}{\partial n'}G_D^0(\bar{r},\bar{r}')ds' - 2\int_A H_z^1(\bar{r}')\frac{\partial}{\partial y'}G^0(\bar{r},\bar{r}')ds', \quad \bar{r} \in D \tag{4.54}$$

where K_t^1 and H_z^1 are the second terms in the low-frequency expansion of the current in the dent and the magnetic field in the aperture. We therefore have the following expansions

$$K_t = 2 + kdK_t^{01} + K_t^{rr}, \quad H_z = 2 + kdH_z^{01} + H_z^{rr} \tag{4.55}$$

where $K_t^{01} = \frac{1}{kd}K_t^1$ and $H_z^{01} = \frac{1}{kd}H_z^1$ are independent of kd. Furthermore, $\frac{1}{kd}K_t^{rr}$ and $\frac{1}{kd}H_z^{rr}$ tend to zero as $kd \to 0$. Thus, the first two terms in the low-frequency expansion of the current in the dent are of order $(kd)^0$ and kd. Because the second term in the expansion of the derivative of the Green's function (see (2.18)) is of order $(kd)^2 \ln kd$ the third term in the expansion of the current will in general be a function of both kd and $\ln kd$. Consequently, the current does not have a power series expansion.

Next we will show that the current K_t^{01} can be found from the solution to the electrostatic problem with the scatterer situated in the impressed electrostatic field

$$\bar{E}^i_{static} = -\sqrt{\frac{\mu}{\epsilon}}\cos\phi^i\hat{y}. \tag{4.56}$$

To do this we introduce a vector potential $\bar{F}^0 = F_z^0\hat{z}$ for the electrostatic solution $\bar{E}^0$ so that

$$\bar{E}^0 = \sqrt{\frac{\mu}{\epsilon}}\nabla \times \bar{F}^0 \tag{4.57}$$

We find that F_z^0 satisfies Laplace's equation and the Neumann boundary condition on the conductor. As we did for the magnetostatic problem, divide the scattered electrostatic field into reflected and diffracted electrostatic fields. The diffracted electrostatic field at infinity behaves as if it were scattered from a cylinder with a finite cross section. This assumption is confirmed by the corresponding electrostatic field in the bump problem (see Chapter 3). Because the total charge on the conductor is zero it follows from [20, sec.3.11] that the potential F_z^{0d} for the diffracted field satisfies $F_z^{0d} \sim |\nabla \ln r| = \frac{1}{r}$ and $\frac{\partial}{\partial r} F_z^{0d} \sim \frac{1}{r^2}$ as $r \to \infty$. Using these facts and the same derivation that led to the time-harmonic equations (4.46) and (4.48) implies that F_z^0 satisfies these two equations with G_D, G, K_t, H_z, and H_z^i replaced by G_D^0, G^0, F_z^0, F_z^0, and F_z^{0i}, respectively. Here F_z^{0i} is the potential for the impressed electrostatic field (4.56). We find that this potential must be given by

$$F_z^{0i}(\bar{r}) = \cos\phi^i x + C \tag{4.58}$$

where C is a constant. The coupled integral equations for $\bar{F}^0$ are thus

$$\int_D F_z^0(\bar{r}')\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')ds' = -F_z^0(\bar{r}) + x\cos\phi^i + C, \quad \bar{r} \in A \tag{4.59}$$

and

$$\frac{1}{2}F_z^0(\bar{r}) = -\int_D F_z^0(\bar{r}')\frac{\partial}{\partial n'}G_D^0(\bar{r},\bar{r}')ds' - 2\int_A F_z^0(\bar{r}')\frac{\partial}{\partial y'}G^0(\bar{r},\bar{r}')ds', \quad \bar{r} \in D. \tag{4.60}$$

Using the divergence theorem and the result of the integration near the singularity [14, App.] one finds

$$\int_D C\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')ds' = -\frac{1}{2}C, \quad \bar{r} \in A \tag{4.61}$$

and

$$\frac{1}{2}C = -\int_D C\frac{\partial}{\partial n'}G_D^0(\bar{r},\bar{r}')ds' - 2\int_A C\frac{\partial}{\partial y'}G^0(\bar{r},\bar{r}')ds', \quad \bar{r} \in D. \tag{4.62}$$

Consequently, the constant C on the right hand side of the integral equation (4.59) only adds the constant C to the solution to (4.59)-(4.60) with $C = 0$. Because a constant may be added to the potential $\bar{F}^0$ without changing the electrostatic solution $\bar{E}^0$, let $C = 0$ in (4.59). By comparing the integral equations (4.53)-(4.54) to (4.59)-(4.60) with $C = 0$ we find that $K_t^{01}(\bar{r}) = \frac{j}{d}F_z^0(\bar{r})$ when $\bar{r} \in D$. This completes the proof that the second term in the low-frequency expansion of the current can be found from the electrostatic solution.

It is convenient to introduce a potential F_z^{0D} which is the solution to the equations

$$\int_D F_z^{0D}(\bar{r}')\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')ds' = -F_z^{0D}(\bar{r}) + \frac{1}{2}x, \quad \bar{r} \in A. \tag{4.63}$$

and

$$\frac{1}{2}F_z^{0D}(\bar{r}) = -\int_D F_z^{0D}(\bar{r}')\frac{\partial}{\partial n'}G_D^0(\bar{r},\bar{r}')ds' - 2\int_A F_z^{0D}(\bar{r}')\frac{\partial}{\partial y'}G^0(\bar{r},\bar{r}')ds', \; \bar{r} \in D \tag{4.64}$$

The low-frequency expansion of the current can now be written in terms of F_z^{0D} as

$$K_t = 2 + 2jk\cos\phi^i F_z^{0D} + K_t^{rr} \tag{4.65}$$

where $\frac{1}{kd}K_t^{rr} \to 0$ as $kd \to 0$.

4.2.3 Low-frequency diffracted far field (TE)

Having derived a low-frequency expansion of the current in the dent in the previous section, we will now integrate this low-frequency expansion to get the low-frequency diffracted far field. We start by deriving a formula that expresses the TE diffracted field in the region F in terms of the current in the dent.

To do this, note that the Neumann Green's function $G_N(\bar{r},\bar{r}')$ satisfies the homogenous Helmholtz equation when $\bar{r} \in F$ and $\bar{r}' \in int(A \cup D)$. Green's second identity therefore gives

$$0 = \int_{A\cup D} [G_N(\bar{r},\bar{r}')\frac{\partial}{\partial n'}H_z(\bar{r}') - H_z(\bar{r}')\frac{\partial}{\partial n'}G_N(\bar{r},\bar{r}')]ds', \quad \bar{r} \in F. \tag{4.66}$$

Because G_N and H_z satisfy the Neumann boundary condition on A and D, respectively, and because $G_N = 2G$ on A, (4.66) implies

$$\int_D H_z(\bar{r}')\frac{\partial}{\partial n'}G_N(\bar{r},\bar{r}')ds' = 2\int_A G(\bar{r},\bar{r}')\frac{\partial}{\partial y'}H_z(\bar{r}')ds', \quad \bar{r} \in F. \tag{4.67}$$

Using this in the field expression (4.40) gives [18]

$$H_z^d(\bar{r}) = -\int_D K_t(\bar{r}')\frac{\partial}{\partial n'}G_N(\bar{r},\bar{r}')ds', \quad \bar{r} \in F \tag{4.68}$$

which is the relation between the diffracted field and the current in the dent.

From the asymptotic expansion of the Hankel function it is found that

$$\begin{aligned}\frac{\partial}{\partial n'}G_N(\bar{r},\bar{r}') \sim k\frac{e^{j\pi/4}}{\sqrt{2\pi}}\frac{e^{-jkr}}{\sqrt{kr}}\hat{n}' \cdot [\hat{x}\cos\phi + \hat{x}jkx'\cos^2\phi \\ +\hat{y}jky'\sin^2\phi + O((kd)^2)], \quad r \to \infty.\end{aligned} \tag{4.69}$$

If the expansion (4.65) of the current and the expansion (4.69) of the Green's function are inserted into the field expression (4.68) one finds that to order $(kd)^2$ the diffracted far field is given by

$$\begin{aligned}H_z^d(\bar{r}) \sim -(kd)^2\frac{e^{j\pi/4}}{\sqrt{2\pi}}\frac{e^{-jkr}}{\sqrt{kr}}\frac{2j}{d^2}[\int_D(\hat{x}x'\cos^2\phi + \hat{y}y'\sin^2\phi)\cdot\hat{n}'ds' \\ +\cos\phi\cos\phi^i\int_D\frac{\partial y'}{\partial s'}F_z^{0D}(\bar{r}')ds'].\end{aligned} \tag{4.70}$$

With the help of the divergence theorem one finds that the first integral in this equation simply equals the area of the region bounded by $A \cup D$, that is, the area of the cross section of the dent. The second integral in (4.70) may be written as

$$\int_D \frac{\partial y'}{\partial s'}F_z^{0D}(\bar{r}')ds' = \int_D y'\hat{n}' \cdot \bar{E}^{0D}(\bar{r}')ds' \tag{4.71}$$

after integrating by parts once and introducing the electrostatic solution $\bar{E}^{0D}$ to the problem in which the scatterer is situated in the impressed electrostatic field $\frac{1}{2}\hat{y}$.

Before writing the final expression for the diffracted far field, we shall prove the relationship

$$\hat{n} \cdot \bar{E}^{0D}(\bar{r}) = K_z^{0D}(\bar{r}), \quad \bar{r} \in D \tag{4.72}$$

between the TE electrostatic solution in (4.71) and the TM magnetostatic solution in (4.28).

To do this introduce a scalar potential ψ so that $\bar{E}^{0D} = -\nabla\psi$ with ψ satisfying Laplace's equation and the Dirichlet boundary condition on the conductor exactly as does the vector potential $\bar{A}_1^0$ in Section 4.1.2. Furthermore, because the total charge on the conductor is zero the potential ψ^d for the diffracted field also behaves like the potential $\bar{A}_1^{0d}$ at infinity [20, sec.3.11]. Therefore, the relations $K_{1z}^0 = \frac{\partial}{\partial n} A_{1z}^0$, $H_{1x}^0 = \frac{\partial}{\partial y} A_{1z}^0$, and $\psi^i = -y + C$ where C is a constant and ψ^i is the potential for the impressed electrostatic field, imply from the coupled integral equations (4.26)-(4.27) that ψ satisfies the coupled integral equations

$$\int_D \frac{\partial}{\partial y'} G^0(\bar{r},\bar{r}') \frac{\partial}{\partial n'} \psi(\bar{r}') ds' = -\frac{1}{2} - \frac{\partial}{\partial y'} \psi(\bar{r}), \quad \bar{r} \in A, \tag{4.73}$$

and

$$⨍_D G_N^0(\bar{r},\bar{r}') \frac{\partial}{\partial n'} \psi(\bar{r}') ds' + 2\int_A G^0(\bar{r},\bar{r}') \frac{\partial}{\partial y'} \psi(\bar{r}') ds' = 0, \quad \bar{r} \in D. \tag{4.74}$$

Using the fact that ψ satisfies Laplace's equation and employing the divergence theorem in the region bounded by $A \cup D$ give

$$\int_{A\cup D} \frac{\partial}{\partial n'} \psi(\bar{r}') ds' = \int_{A\cup D} \hat{n}' \cdot \nabla\psi(\bar{r}') ds' = \int_{int(A\cup D)} \nabla^2 \psi(\bar{r}') ds' = 0. \tag{4.75}$$

Using the relation $\hat{n} \cdot \bar{E}^{0D}(\bar{r}) = -\frac{\partial}{\partial n}\psi(\bar{r}), \quad \bar{r} \in D$, and comparing (4.73), (4.74), and (4.75) to (4.29), (4.31), and (4.30), respectively, shows that (4.72) indeed is a correct identity.

According to the relations (4.71) and (4.72), the diffracted far field in (4.70) may be written as

$$H_z^d(\bar{r}) \sim (kd)^2 e^{-j\pi/4} \sqrt{\frac{2}{\pi}} \frac{e^{-jkr}}{\sqrt{kr}} \frac{1}{d^2} [A_D + D_0 \cos\phi \cos\phi^i] \tag{4.76}$$

where A_D is the area of the cross section of the dent and

$$D_0 = \int_D K_z^{0D}(\bar{r}') y' ds' = \int_D F_z^{0D}(\bar{r}') \frac{\partial y'}{\partial s'} ds'. \tag{4.77}$$

The magnetostatic current $\bar{K}^{0D}$ is found from the coupled integral equations (4.29), (4.30), and (4.31) and the electrostatic vector potential $\bar{F}^{0D}$ is found from the coupled integral equations (4.63) and (4.64).

The diffracted far field in (4.76) is the contribution from a magnetic dipole in the $z-$direction (the term with A_D) and an electric dipole in the $y-$direction (the cosine term) [11, sec.3.8]. Furthermore, the constant in the TE electric dipole term in (4.76) is the same as in the TM magnetic dipole term (4.36).

We have now shown that the low-frequency scattering from a two-dimensional dent in a ground plane, as for bump scattering, reduces to the evaluation of a single constant D_0.

This constant is determined from a static solution and depends only on the shape of the cross section of the dent.

Throughout the derivation that led to (4.76) it was assumed that the dent was continuously lined by a conductor so that our results in this paper do not apply to the slit in a ground plane. However, since the slit is the complement of the strip it is seen from Chapter 2 that the low-frequency fields diffracted by a slit for TE polarization must be of order $\frac{1}{\ln kd}$ where d is the width of the slit. Thus, the low-frequency fields diffracted by slits and dents are of different order and, in particular, the slit scatters more strongly than the dent at low frequencies. The physical reason for this enhanced scattering is that the slit completely "stops" the currents on the ground plane while the dent only "diverts" them.

Chapter 5

Evaluation of constants and verification of low-frequency expressions

This section evaluates numerically the constants occurring in the low-frequency far-field expressions for particular geometries. Furthermore, the expressions will be verified by comparing them to low-frequency results obtained from exact time-harmonic eigenfunction solutions when such solutions exist.

5.1 Circular cylinder

Begin by considering the circular cylinder with radius a situated with its center at the origin of the coordinate system. From the exact eigenfunction solution it is found that the low-frequency expression for the scattered far fields are [7, sec.2.2.1.2, sec.2.2.2.2]

$$TM: \qquad E_z^s(\bar{r}) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-j\pi/4}}{\ln ka} \frac{e^{-jkr}}{\sqrt{kr}} \tag{5.1}$$

$$TE: \qquad H_z^s(\bar{r}) \sim -(ka)^2 \frac{e^{-j\pi/4}}{2\sqrt{2\pi}} \frac{e^{-jkr}}{\sqrt{kr}} [\pi + 2\pi \sin\phi \sin\phi^i + 2\pi \cos\phi \cos\phi^i]. \tag{5.2}$$

We see that (5.1) and (5.2) agree with the general results in (2.12) and (2.38) provided the three constants in (2.39) are

$$\frac{C_1}{a^2} = \frac{C_2}{a^2} = 2\pi, \quad C_3 = 0. \tag{5.3}$$

We can check the expressions (?.39) for the three constants in two ways.

The first way is to find exact expressions for the electrostatic charges σ^{0x} and σ^{0y} using the method of separation of variables and then insert these into (2.39). It is easily found

that [21, pp.1184-1185]

$$\sigma^{0x} = 2\epsilon \cos\phi, \quad \sigma^{0y} = 2\epsilon \sin\phi \tag{5.4}$$

and by inserting these into (2.39) one recovers (5.3).

The second way of verifying the constants is to solve the integral equations (2.42) and (2.43) numerically and then use (2.47) to evaluate the constants. These two integral equations were solved numerically by using pulse expansion functions and point matching. The moment matrix elements were calculated using Simpson's rule of integration. The following values of the constants were obtained with 400 pulse expansion functions

$$\frac{C_1}{a^2} = \frac{C_2}{a^2} = 6.283069, \quad C_3 = 0.4 \cdot 10^{-14} \tag{5.5}$$

which verify our general low-frequency results for scattering from cylinders with finite cross section. We also solved the integral equations (2.40) and (2.41) for σ^{0x} and σ^{0y} and inserted these numerical solutions into (2.39). Again the result agreed with (5.3).

5.2 Semi-circular bump

Next consider the semi-circular bump of radius a. Rayleigh [17] used the exact eigenfunction solution for the circular cylinder to show that

$$TM: \quad E_z^d(\bar{r}) \sim (ka)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{e^{-jkr}}{\sqrt{kr}}\pi \sin\phi \sin\phi^i \tag{5.6}$$

$$TE: \quad H_z^d(\bar{r}) \sim -(ka)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{e^{-jkr}}{\sqrt{kr}}[\frac{\pi}{2} + \pi\cos\phi\cos\phi^i]. \tag{5.7}$$

We see that this agrees with the general low-frequency expressions (3.13) and (3.19) provided that

$$\frac{B_0}{a^2} = -\pi. \tag{5.8}$$

Again from the method of separation of variables, $\sigma^{0y} = 2\epsilon\sin\phi$ and one finds that $K_z^{0B} = -2\sin\phi$. Inserting these functions into (3.14) one recovers the value of B_0 given in (5.8). One may also combine the numerical results for the circular cylinder obtained by numerically solving the integral equation (2.42), with the relation $B_0 = -\frac{1}{2}C_2$ where C_2 is given in (5.3). Alternatively, we can solve the simple integral equation (3.9) for the current K_z^{0B}. This was done using pulse expansion functions and point matching. The following value of B_0 was obtained with 400 expansion functions

$$\frac{B_0}{a^2} = -3.1415965 \tag{5.9}$$

which confirms our general low-frequency expression for the diffracted far fields of a cylindrical bump.

5.3 Knife-edge bump

Consider the vertical knife-edge bump with height h. Using the exact time-harmonic eigenfunction solution for the strip [7, ch.4] and Rayleigh's method of constructing an equivalent scattering problem, it is found that the low-frequency expressions for the knife-edge bump are

$$TM: \quad E_z^d(\bar{r}) \sim (kh)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{e^{-jkr}}{\sqrt{kr}}\frac{\pi}{2}\sin\phi\sin\phi^i \tag{5.10}$$

$$TE: \quad H_z^d(\bar{r}) \sim -(kh)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{e^{-jkr}}{\sqrt{kr}}\frac{\pi}{2}\cos\phi\cos\phi^i. \tag{5.11}$$

We see that these expressions agree with the general expressions (3.13) and (3.19) provided that

$$\frac{B_0}{h^2} = -\frac{\pi}{2}. \tag{5.12}$$

Again we shall verify that (3.14) gives this value for B_0 in two different ways. First we will find the static current K_z^{0B} using separation of variables in elliptic coordinates and second we will find K_z^{0B} by solving the simple integral equation (3.9) numerically.

To determine K_z^{0B} from the method of separation of variables it is convenient to use the results in Morse and Feshbach [21, pp.1195-1197]. K_z^{0B} is the static current in that magnetostatic problem in which a strip of width $2h$, which has its cross section directed along the y-axis, is situated in the magnetostatic field $\hat{x}$. Therefore, we define the elliptic coordinates (μ, θ) by

$$x = -h\sinh\mu\sin\theta, \quad y = h\cosh\mu\cos\theta \tag{5.13}$$

where $\mu \geq 0$ and $0 \leq \theta \leq 2\pi$. The θ-coordinate and the strip given by $\mu = 0$ are shown in Figure 5.1. The scattering problem for K_z^{0B} can be solved using a vector potential $\bar{A}^0 = \hat{z}A_z^0$ for the magnetostatic field. The impressed magnetostatic field can be represented by the impressed vector potential

$$\bar{A}^{0i} = y\hat{z} = h\cosh\mu\cos\theta\hat{z}. \tag{5.14}$$

The total vector potential $\bar{A}^0$ must be constant on the strip $\mu = 0$ and the vector potential $\bar{A}^{0s}$ for the scattered field must tend to zero at infinity. Therefore $\bar{A}^{0s}$ is given in terms of the elementary solution $e^{-\mu}\cos\theta$ [21, p.1195], that is,

$$\bar{A}^{0s} = Ce^{-\mu}\cos\theta\hat{z} \tag{5.15}$$

where C is a constant. Requiring that the vector potential $\bar{A}^0$ for the total magnetostatic field be zero on the strip implies that $C = -h$, that is,

$$\bar{A}^0 = h\sinh\mu\cos\theta\hat{z} = -x\frac{\cos\theta}{\sin\theta}\hat{z}. \tag{5.16}$$

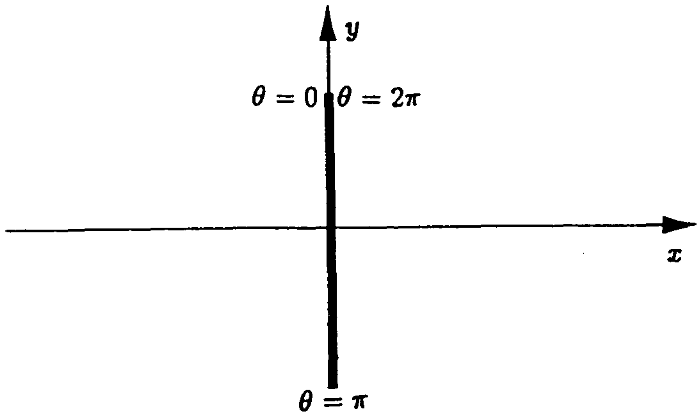

Figure 5.1: Strip of width $2h$ in an elliptic coordinate system.

Consequently, the current K_z^{0B} is

$$K_z^{0B}(\theta) = \begin{cases} -\frac{\cos\theta}{\sin\theta} & 0 < \theta < \pi \\ +\frac{\cos\theta}{\sin\theta} & \pi < \theta < 2\pi \end{cases} \tag{5.17}$$

The constant B_0 in (3.14) is

$$B_0 = \frac{1}{2}\int_{\theta=0}^{2\pi} y K_z^{0B}(\bar{r})ds = -\frac{1}{2}h^2 \int_0^{2\pi} \cos^2\theta d\theta = -\frac{\pi}{2}h^2 \tag{5.18}$$

which agrees with (5.12).

Another verification is obtained by solving the integral equation (3.9) numerically and then integrating the numerical solution to get B_0. The following value of B_0 was obtained with 400 pulse expansion functions

$$\frac{B_0}{h^2} = -1.5706168 \tag{5.19}$$

which again verifies the general low-frequency expressions for the diffraction from cylindrical bumps.

5.4 Right-angled cylinders and bumps

We now present numerically evaluated constants for the low-frequency expressions in cases where no exact time-harmonic eigenfunction solutions exist.

Using the integral equations (2.42) and (2.43) with 400 pulse expansion functions it was found that

$$\frac{C_1}{d^2} = \frac{C_2}{d^2} = 8.7596, \quad C_3 = 0 \tag{5.20}$$

for a square cylinder with side length $2d$ and sides parallel to the x and y axes. It was furthermore found that the constants changed to

$$\frac{C_1}{d^2} = 13.3361, \ \frac{C_2}{d^2} = 25.1541, \ C_3 = 0 \tag{5.21}$$

for a rectangular cylinder with a $4d$ side length parallel to $\hat{y}$ and a $2d$ side length parallel to $\hat{x}$.

The following bump constants were obtained by solving numerically the simple integral equation (3.9) using 400 pulse expansion functions. For the square bump with side length $2d$ it was found that

$$\frac{B_0}{d^2} = -12.5666 \tag{5.22}$$

and for the right-angled triangular bump with height $\sqrt{2}d$ we computed

$$\frac{B_0}{d^2} = -2.1881684. \tag{5.23}$$

5.5 Semi-circular dent

Consider the semi-circular dent with radius a situated with its center at the origin of the coordinate system. The dual-series eigenfunction solution [22] shows that to order $(ka)^2$

$$TM: \quad E_z^d(\bar{r}) \sim -(ka)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{e^{-jkr}}{\sqrt{kr}}(0.58)\sin\phi\sin\phi^i \tag{5.24}$$

$$TE: \quad H_z^d(\bar{r}) \sim (ka)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{e^{-jkr}}{\sqrt{kr}}[\frac{\pi}{2} + (0.58)\cos\phi\cos\phi^i]. \tag{5.25}$$

These expressions agree with our general result in (4.36) and (4.76) provided that

$$\frac{D_0}{a^2} = 0.58. \tag{5.26}$$

To determine the value of D_0 we solved numerically the coupled integral equations (4.29), (4.30), and (4.31) determining the magnetostatic current K_z^{0D}. In the numerical solution we used pulse expansion functions and point matching and the equation (4.30) was incorporated by simply adding it to the linear equations obtained from (4.29) and (4.31). Simpson's rule of integration was used to evaluate the moment matrix elements. Using 150 pulse expansion functions for both the current in the dent and the magnetic field in the aperture (300 unknowns in total) we found using (4.37) to determine D_0 that

$$\frac{D_0}{a^2} = 0.58236. \tag{5.27}$$

We also solved numerically the coupled integral equations (4.63) and (4.64) for the electrostatic vector potential F_z^{0D}. These equations were also solved using pulse expansion functions and point matching. Again Simpson's rule of integration was used to evaluate the moment method matrix. Using 225 pulse expansion functions for both the potential in the dent and the potential in the aperture (450 unknowns in total) we found using (4.77) that

$$\frac{D_0}{a^2} = 0.58238. \tag{5.28}$$

Comparing the values of D_0 obtained solving static integral equations with the one obtained from the dual-series eigenfunction solution [22] confirms our general low-frequency results.

5.6 Rectangular dents

Finally some results will be given for the constant D_0 in the case of rectangular dents.

The following results were found using the integral equations (4.63) and (4.64) with 225 pulse expansion functions for both the potential in the dent and aperture (450 unknowns in total). For the rectangular dent with depth $4d$, $2d$, d, $0.5d$, and $0.1d$ we found that $\frac{D_0}{d^2}$ equals 0.63320, 0.63304, 0.60591, 0.49679, and 0.16790, respectively. The length d in is shown in Figure 4.1.

Chapter 6

Conclusion

We have evaluated the low-frequency electromagnetic scattering from the perfectly conducting cylinder with finite cross section and the perfectly conducting cylindrical bump and dent in a ground plane.

For the cylinder with finite cross section the low-frequency scattered far field for TM polarization is independent of the shape of the cross section of the cylinder and of order $\frac{1}{\ln kd}$ where d is a characteristic dimension of the cylinder. This low-frequency result is not related to a corresponding static field problem.

For TE polarization the scattered field is of order $(kd)^2$ and it consists of a contribution from a magnetic dipole along the axis of the cylinder and an electric dipole in a direction normal to the axis of the cylinder. The magnetic dipole moment is found directly from the area of the cross section of the cylinder. The electric dipole moment is found by solving an electrostatic problem, that is, a two-dimensional potential problem, for two impressed fields and integrating these two electrostatic solutions around the cylinder. These electrostatic solutions are determined from simple static integral equations and depend only on the shape of the cylinder.

For both the cylindrical bump on a ground plane and dent in a ground plane the low-frequency diffracted field for TM and TE polarization is of order $(kd)^2$, where d is a characteristic dimension of the bump or dent. The low-frequency TM diffracted far field is that of a magnetic dipole normal to the axial direction and parallel to the ground plane.

The low-frequency TE diffracted far field for both the cylindrical bump and dent consists of a contribution from a magnetic dipole in the axial direction and an electric dipole normal to both the axial direction and the ground plane. The TE magnetic dipole moment is found directly from the area of the cross section of the bump or dent.

Both the TM magnetic dipole field and the TE electric dipole field can be written as a constant times a known simple function. It is proven that, remarkably, this constant, which depends only on the shape of the bump or dent, is the same for both the TM and TE polarizations. This constant can be found by solving either a magnetostatic or electrostatic field problem, that is, a two-dimensional potential problem, and performing an integration of these solutions over the bump or dent. This means that both the TM and TE low-frequency

diffracted far fields for an arbitrarily shaped bump or dent are completely determined by calculating a single constant for the bump or dent.

The low-frequency expressions were confirmed from exact time-harmonic eigenfunction solutions, and constants were evaluated for a number of geometries. As mentioned in the Introduction, the low-frequency expressions derived in this report are given in closed form and can therefore be used directly to determine incremental length diffraction coefficients for calculating the scattered fields from curved narrow ridges and channels in conductors [9]. These incremental length diffraction coefficients determined from the low-frequency expressions derived in this report have been applied recently to calculate the effects of ridges and channels on the fields of a reflector antenna [23].

References

[1] Lord Rayleigh, On the incidence of areal and electric waves upon small obstacles in the form of ellipsoids or elliptic cylinders, and on the passage of electrical wave through a circular aperture in a conducting screen, *Phil. Mag.*, **44**: 28-52, 1897.

[2] R.E.Kleinman, The Rayleigh region, *Proc. IEEE*, **53**: 848-856, 1965.

[3] A.F.Stevenson, Solution of electromagnetic scattering problems as power series in the ratio (dimension of scatterer)/wavelength, *J.Appl. Phys.*, **24**: 1134-1142, 1953.

[4] J.S.Asvestas and R.E.Kleinman, Low-frequency scattering by perfectly conducting obstacles, *J.Math.Phys.*, **12**: 795-811, May 1971.

[5] J.van Bladel, Low-frequency scattering by cylindrical bodies, *Appl. Sci.Res.*, Sec.B, **10**: 195-202, 1963. Also in Sec. 12.6 of van Bladel's book, *Electromagnetic Fields*, New York, McGraw-Hill, 1964.

[6] R.C.MacCamy, Low-frequency acoustic oscillations,*J.Quart. Appl. Math.*, **23**: 247-255, October 1965.

[7] J.J. Bowman, T.B.A. Senior, and P.L.E. Uslenghi, *Electromagnetic and Acoustic Scattering by Simple Shapes*, Amsterdam : North-Holland, 1969.

[8] R.A.Shore and A.D.Yaghjian, Incremental diffraction coefficients for planar surfaces, *IEEE Trans. Antennas Propagat.*, **36**: 55-70, January 1988.

[9] T.B.Hansen and A.D.Yaghjian, Incremental diffraction coefficients for cylinders of arbitrary cross section: application to diffraction from ridges and channels in perfectly conducting surfaces, *IEEE AP-Symposium Digest*, June 24-28, 1991, The Univ. of Western Ontario, London, Ontario, Canada: 794-797.

[10] V.Twersky, On scattering and reflection of electromagnetic waves by rough surfaces, *IEEE Trans. Antennas Propagat.*, **5**: 81-90, 1957.

[11] D.S. Jones, *The Theory of Electromagnetism*, New York: MacMillan, 1964.

[12] T.B.Hansen and A.D.Yaghjian, Low-frequency scattering from two-dimensional perfect conductors, *IEEE AP-Symposium Digest*, June 24-28, 1991, The Univ. of Western Ontario, London, Ontario, Canada: 798-801.

[13] C.H.Papas, Diffraction by a cylindrical obstacle, *J.Appl.Phys.*, **21**: 318-325, 1950.

[14] K.K.Mei and J.G.van Bladel, Scattering by perfectly-conducting rectangular cylinders, *IEEE Trans. Antennas Propagat.*, **11**: 185-192, 1963.

[15] T.B.Hansen, Uniqueness theorems for static integral equations and calculation of constants for 2-D low-frequency scattering, To be published.

[16] A.J.Poggio and E.K.Miller, Integral equation solutions of three-dimensional scattering problems, in *Computer Techniques for Electromagnetics*, R.Mittra, ed., New York: Pergamon, 1973: 159-264.

[17] Lord Rayleigh, On the light dispersed from fine lines ruled upon reflecting surfaces or transmitted by very narrow slits, *Phil. Mag.*, **14**: 350-359, 1907.

[18] J.S.Asvestas and R.E.Kleinman, *Electromagnetic scattering by indented screens*, University of Delaware, Department of Mathematical Sciences, Technical Report No. 92-6, 1992.

[19] T.B.Hansen and A.D.Yaghjian, Low-frequency scattering from two-dimensional perfect conductors, Accepted for publication in *IEEE Trans. Antennas Propagat.*

[20] J.A.Stratton, *Electromagnetic Theory*, New York: McGraw-Hill, 1941.

[21] P.M.Morse and H.Feshbach, *Methods of Theoretical Physics*, New York: McGraw-Hill, 1953, Part II.

[22] M.K.Hinders and A.D.Yaghjian, Dual-series solution to scattering from a semicircular channel in a ground plane, *IEEE Microwave and Guided Wave Letters*, **1**: 239-242, September 1991. See also M.K.Hinders, *Scattering of a plane electromagnetic wave from a semicircular crack in a perfectly conducting ground plane*, RADC Tech. Rep., TR-89-12, April 1989. **ADA219310**

[23] R.A.Shore and A.D.Yaghjian, Application of incremental length diffraction coefficients to calculate the effects of rim and surface cracks of a reflector antenna, in review for publication in *IEEE Trans. Antennas Propagat.*

[24] H.B.Phillips, *Vector Analysis*, New York: Wiley and Sons, 1933.

Appendix A

Existence of an electrostatic vector potential

Let $\bar{E}^0$ be the total electrostatic field in the TE electrostatic problem in which the scatterer is the cylinder with finite cross section shown in Figure 2.1. From Maxwell's equations we find that $\bar{E}^0$ has zero divergence, that is,

$$\nabla \bar{E}^0 = 0 \tag{A.1}$$

and because the total charge on the conductor is zero we have

$$\int_S \bar{E}^0 \cdot \hat{n} ds = 0 \tag{A.2}$$

where S is the curve that describes the cross section of the cylinder.

We will show that (A.1) and (A.2) imply that there exists an electrostatic vector potential

$$\bar{F}^0 = F_z^0 \hat{z} \tag{A.3}$$

so that

$$\bar{E}^0 = \nabla \times \bar{F}^0. \tag{A.4}$$

We start by proving that

$$\int_C \bar{E}^0 \cdot \hat{n} ds = 0 \tag{A.5}$$

for every closed curve C in the free-space region of the $x - y$ plane of Figure 2.1. Here $\hat{n} = \hat{t} \times \hat{z}$ where $\hat{t}$ is the tangent unit vector to C.

Consider first the case where C does not enclose the cylinder S: Then, since the electrostatic field has zero divergence it follows from the divergence theorem that (A.5) is satisfied.

Consider then the case where C encloses S: The divergence theorem shows that the integral over C equals plus or minus the integral over S, which by (A.2) is zero.

We have now shown that the integral over every closed free-space curve C of $\hat{n} \cdot \bar{E}^0$ is zero.

Let $(E_x^0, E_y^0, 0)$ be the rectangular components of the electrostatic field $\bar{E}^0$, then we have the following identity on C

$$\hat{n} \cdot \bar{E}^0 = \hat{t} \cdot [-E_y^0 \hat{x} + E_x^0 \hat{y}]. \tag{A.6}$$

Because of (A.5) we know that the field $-E_y^0 \hat{x} + E_x^0 \hat{y}$ is conservative and it follows then from Phillips [24, p.52] that there exists a scalar potential ψ such that

$$-E_y^0 \hat{x} + E_x^0 \hat{y} = \nabla \psi. \tag{A.7}$$

By inspection one finds that (A.4) holds with

$$F_z^0 = \psi \tag{A.8}$$

and the existence of a vector potential for the electrostatic field $\bar{E}^0$, satisfying (A.1) and (A.2), is proved.

Appendix B

Dipole moment reciprocity theorem

Consider the two electrostatic problems where the cylinder in Figure 2.1 is situated in the impressed electrostatic fields $\bar{E}_1^i = \hat{x}$ and $\bar{E}_2^i = \hat{y}$. Let the solution to these two static problems be

$$\bar{E}_{1,2}^0 = \bar{E}_{1,2}^i + \bar{E}_{1,2}^s \tag{B.1}$$

where $\bar{E}_{1,2}^s$ are the scattered fields.

We will prove the dipole moment reciprocity theorem:

$$\int_S y\hat{n} \cdot \bar{E}_1^0 ds = \int_S x\hat{n} \cdot \bar{E}_2^0 ds \tag{B.2}$$

which states that the y component of the dipole moment in the scattering problem with $\bar{E}^i = \hat{x}$ equals the x component of the dipole moment in the scattering problem with $\bar{E}^i = \hat{y}$. To prove this we introduce the scalar potentials $\psi_{1,2}$ such that

$$\bar{E}_{1,2}^0 = -\nabla\psi_{1,2}. \tag{B.3}$$

The potentials $\psi_{1,2}^i$ for the impressed fields are given by

$$\psi_1^i = -x + C_1 \tag{B.4}$$

and

$$\psi_2^i = -y + C_2 \tag{B.5}$$

where C_1 and C_2 are constants. Since the total charge on the conductor is zero, that is,

$$\int_S \hat{n} \cdot \bar{E}_{1,2}^0 ds = 0 \tag{B.6}$$

it is seen that (B.2) is equivalent to

$$\int_S \psi_2^i \frac{\partial\psi_1}{\partial n} ds = \int_S \psi_1^i \frac{\partial\psi_2}{\partial n} ds. \tag{B.7}$$

Because the potentials $\psi^i_{1,2}$ satisfy Laplace's equation inside S we obtain from Green's second identity

$$\int_S \psi^i_2 \frac{\partial \psi^i_1}{\partial n} ds = \int_S \psi^i_1 \frac{\partial \psi^i_2}{\partial n} ds. \tag{B.8}$$

If we require that $\psi_{1,2} = 0$ on S we see from (B.8) that (B.7) and therefore also (B.2) is equivalent to

$$\int_S \psi^s_2 \frac{\partial \psi^s_1}{\partial n} ds = \int_S \psi^s_1 \frac{\partial \psi^s_2}{\partial n} ds \tag{B.9}$$

where $\psi^s_{1,2} = \psi_{1,2} - \psi^i_{1,2}$ are the potentials for the scattered electrostatic fields.

Let Σ_R be the circle with radius R and center at $x = y = 0$. Assume that Σ_R encloses S. Then since $\psi^s_{1,2}$ satisfy Laplace's equation outside S, we get from Green's second identity that

$$0 = \int_S \left[\frac{\partial \psi^s_1}{\partial n} \psi^s_2 - \frac{\partial \psi^s_2}{\partial n} \psi^s_1 \right] ds - \int_{\Sigma_R} \left[\frac{\partial \psi^s_1}{\partial r} \psi^s_2 - \frac{\partial \psi^s_2}{\partial r} \psi^s_1 \right] ds. \tag{B.10}$$

Now, the total charge is zero, so it follows from [20, sec.3.11] that for two-dimensional problems

$$\psi^s_{1,2} \sim |\nabla \ln r| = \frac{1}{r}, \quad r \to \infty \tag{B.11}$$

and

$$\frac{\partial \psi^s_{1,2}}{\partial r} \sim \frac{1}{r^2}, \quad r \to \infty. \tag{B.12}$$

The asymptotic conditions (B.11) and (B.12) imply that the last integral in (B.10) goes to zero as R goes to infinity. Thus, (B.9) and therefore the dipole moment reciprocity theorem is proven.